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5 Propagation of Gravitational Waves

In this chapter we shall study the propagation of gravitational waves from their source to the earth. We begin in Sec. 5.A by writing down the propagation laws in their simplest form, that appropriate to the *geometric-optics approximation*. Then in Secs. 5.B, 5.C, and 5.D we derive those geometric-optics propagation laws in a careful manner, identifying along the way the various assumptions that must be made. Each assumption entails discarding physical effects that might be important in special but unusual situations.

For simplicity, our geometric-optics laws are specialized to propagation through vacuum; however, as part of our derivation of them, we obtain in Secs. 5.B and 5.C a propagation equation that describes the interaction of the waves with matter and with electromagnetic fields. In Secs. 5.E and 5.F we seek insight into that interaction by calculations with this propagation equation. Our calculations show that, although the coupling of waves to matter and electromagnetism is fascinating in principle, it is almost never significant in practice: In realistic astrophysical situations the vacuum approximation to wave propagation is excellent.

In Sec. 5.G we study a wide variety of vacuum propagation phenomena (scattering and parametric amplification by background curvature, tails of waves, gravitational focusing and diffraction, nonlinear wave-wave coupling, ...); we describe how to analyze these effects; and we discuss their relevance to wave propagation in the real universe. We conclude the chapter in Sec. 5.H with a brief discussion of two special, non-geometric-optics analyses of wave propagation: a set of exact solutions to the Einstein field equation, which describe propagating waves; and the theory of the asymptotic structure of the gravitational-wave field outside an isolated source in an asymptotically flat spacetime.

In order to understand this chapter and Chap. 6, the reader will need prior familiarity with general relativity at, e.g., the level of "track one" of MTW (Misner, Thorne, and Wheeler, 1973). Readers without such familiarity can move on to Chaps. 7–12, which should be understandable without mastery of Chaps. 5 and 6.

Our notation and mathematical conventions will be those of MTW. A key role will be played, in the mathematical formalism, by a split of the full, gravitational-wave-endowed spacetime into a background spacetime [obtained by averaging over several wavelengths of the waves, cf. Eqs. (4.4) above and (5.6) below], plus the waves. After the split has been made, the waves will be thought of as a field that propagates through the background spacetime. We shall use a vertical slash to denote covariant derivatives, i.e., gradients, in the background spacetime, so if S^α (also denoted abstractly as $\mathbf{S}$) is a vector field that lives in the background spacetime, then ${S^\alpha}_{|\mu}$ (also denoted abstractly as $\nabla\mathbf{S}$) is its gradient.

Similarly, we shall use a semicolon to denote covariant derivatives in the full, pre-split, wave-endowed spacetime, so if A^α lives in the full spacetime, then $A^\alpha{}_{;\mu}$ is its gradient.

5.A. Geometric-optics propagation laws

Geometric optics is a very general formalism for studying the propagation of any kind of wave through any kind of medium. This formalism is valid whenever the wave's reduced wavelength $\bar{\lambda}$ is small compared to the radius of curvature of its wave fronts, and also small compared to all inhomogeneity scales of the medium through which it propagates.

For gravitational waves *in vacuum*, the geometric-optics propagation can be described as follows. (We shall derive this description in Secs. 5.B, 5.C, 5.D, and we shall show in Secs. 5.E and 5.F that it is also valid to high accuracy for gravitational waves propagating through astrophysically realistic matter.)

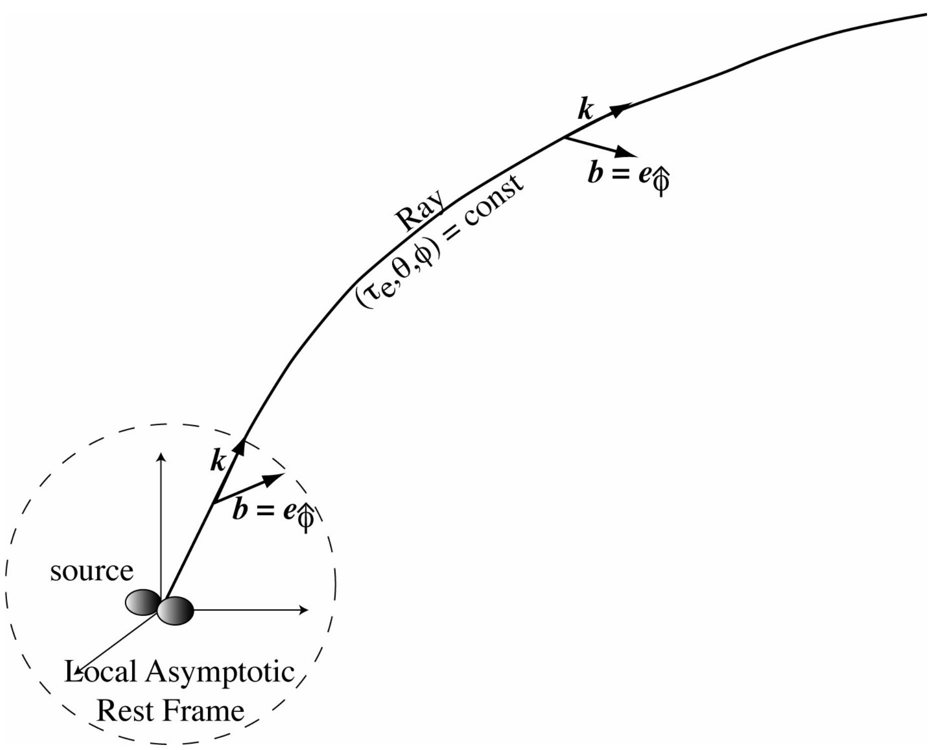

Fig. 5.1 Geometric-optics construction for the propagation of gravitational waves from a source's local asymptotic rest frame (region inside dotted circle) out through the external universe (region outside dotted circle).

Consider, for concreteness, a source of gravitational waves somewhere far out in the universe. In the vicinity of the source but some wavelengths away from it (so as to avoid near-zone fields that will be discussed in Chap. 6), introduce a local Lorentz reference frame in which the source is at rest (the source's *local asymptotic rest frame*; Fig. 5.1). In that frame construct spherical polar coordinates (t, r, θ, ϕ) centered on the source and construct the associated orthonormal basis vectors $\mathbf{e}_0$, $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$. The gravitational waves propagate radially through this asymptotic rest frame, with their retarded time τ_e and their wave vector $\mathbf{k} = -\nabla\tau_e$ (introduced in Sec. 4.D) given by

$$\tau_e = t - r \ , \qquad \mathbf{k} = \mathbf{e}_0 + \mathbf{e}_r \ . \tag{5.1a}$$

The gravitational-wave field $h^{\rm GW}_{\alpha\beta}$ associated with the asymptotic rest frame will have the form [Eq. (4.26)]

$$h^{\rm GW}_{\alpha\beta} = h_+(a_\alpha a_\beta - b_\alpha b_\beta) + h_\times(a_\alpha b_\beta + b_\alpha a_\beta) \ , \tag{5.1b}$$

Here a_α and b_α are polarization vectors that are purely spatial in the source's asymptotic rest frame, have unit length, and are orthogonal to each other and to the waves' (radial) propagation direction. For example, we are free to choose them to be the unit vectors in the θ and ϕ directions

$$\mathbf{a} = \mathbf{e}_{\hat\theta} \ , \qquad \mathbf{b} = \mathbf{e}_{\hat\phi} \ . \tag{5.1c}$$

Any other choice will be rotated relative to this choice by some angle ψ in the transverse plane, and correspondingly h_+ and $h_\times$ will be modified in the manner of expression (4.23). The propagation of the waves (5.1b) through the source's asymptotic rest frame and then on out through the entire universe is described by the following geometric-optics construction:

(i) The waves propagate along a family of null geodesics of the background spacetime, which are called the waves' *rays*. These rays are the specific solutions to the geodesic equation

$$k_{\alpha|\beta}k^\beta = 0 \ , \tag{5.2a}$$

which begin in the asymptotic rest frame with $\mathbf{k} = \mathbf{e}_0 + \mathbf{e}_r$ and extend from there on out through the universe (Fig. 5.1). Each ray is labeled by the direction (θ, ϕ) in which it emerges from the source, and by the retarded time τ_e at which it emerges. By this labeling, τ_e, θ, and ϕ are carried out through the universe on the rays. The tangent vector $\mathbf{k}$ to the rays is given by

$$\mathbf{k} = -\nabla\tau_e \ , \tag{5.2b}$$

not just in the source's local asymptotic rest frame, but everywhere in the universe, as one can verify by checking that this $\mathbf{k}$ satisfies the geodesic equation (5.2a). [Specifically, $k_{\alpha|\beta}k^\beta = \tau_{e|\alpha\beta}\tau_e{}^{|\beta}$, which, by commutation of the covariant derivatives of a scalar field, is equal to $\tau_{e|\beta\alpha}\tau_e{}^{|\beta} = 1/2(\tau_{e|\beta}\tau_e{}^{|\beta})_{|\alpha}$, which vanishes because $\tau_{e|\beta}$ is null.] (ii) Next, parallel transport the polarization vectors a_α and b_α along the waves' rays out through the universe,

$$a_{\alpha|\beta}k^\beta = 0 \ , \qquad b_{\alpha|\beta}k^\beta = 0 \ . \tag{5.2c}$$

(iii) Define all along each ray a *radius function* r in the following way: Consider the 2-dimensional bundle of rays, surrounding the ray of interest, which all have the same τ_e as that ray but which have θ in the range $\Delta\theta$ and ϕ in the range $\Delta\phi$, so they subtend, as seen from the source, a solid angle $\Delta\Omega = \sin\theta\Delta\theta\Delta\phi$. As one moves out along the ray of interest, the cross-sectional area ΔA of this bundle (which is locally Lorentz-invariant; cf. Exercise 22.13 of MTW) changes, but by definition $\Delta\Omega$ remains fixed. The radius function is defined to be

$$r \equiv (\Delta A/\Delta\Omega)^{1/2} \ . \tag{5.2d}$$

Of course, in the source's local asymptotic rest frame this is just equal to the distance to the source. However, far out in the universe it might be quite different from that distance. For example, gravitational lenses (to be discussed in Sec. 5.G.c below) can make the area of the bundle and thence also r switch over from increasing along a ray to decreasing. (*iv*) The (frame-invariant) gravitational-wave fields h_+ and $h_\times$ are carried outward along each ray unchanged, except for an overall alteration proportional to $1/r$ (which is required to conserve the waves' energy as they propagate):

$$h_+ = \frac{Q_+(\tau_e, \theta, \phi)}{r}, \quad h_\times = \frac{Q_\times(\tau_e, \theta, \phi)}{r} . \tag{5.2e}$$

The functions Q_+ and $Q_\times$ can be evaluated in the source's local asymptotic rest frame using the theory of gravitational-wave generation (Chap. 6), and then can be carried out along the rays unchanged. (*v*) The polarization tensors e^+_{jk} and $e^\times_{jk}$ associated with these fields, as measured in any proper reference frame anywhere in the universe, are

$$e^+_{jk} = (a_j a_k - b_j b_k)^{\rm TT} , \quad e^\times_{jk} = (a_j b_k + b_j a_k)^{\rm TT} , \tag{5.2f}$$

where the superscripts TT denote the transverse-traceless projection of Eq. (4.50). (*vi*) Correspondingly, the gravitational-wave field

$$h^{\rm GW}_{jk} = h_+ e^+_{jk} + h_\times e^\times_{jk} , \tag{5.2g}$$

as measured in that proper reference frame, is given by

$$h^{\rm GW}_{jk} = (h_{jk})^{\rm TT} , \tag{5.2h}$$

[Eq. (4.45)], where h_{jk} is the spatial part of the field

$$h_{\alpha\beta} \equiv h_+(a_\alpha a_\beta - b_\alpha b_\beta) + h_\times(a_\alpha b_\beta + b_\alpha a_\beta) . \tag{5.2i}$$

As a simple example, consider the propagation of gravitational waves through a closed Friedman model for our universe. The background metric [obtained by averaging over the waves' short-wavelength ripples in the manner of Eq. (4.4a)] has the standard Friedman form [MTW, Eqs. (27.46) and (27.23)]

$$ds^2 = a^2[-d\eta^2 + d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)] , \tag{5.3}$$

where $a = a(\eta)$ is the universe's "expansion factor". (Never mind that this spacetime is filled with matter rather than being vacuum; as we shall see in Sec. 5.E the presence of matter has no significant influence on the propagation.) Place the source of gravitational waves at the origin of the spatial coordinates, $\chi = 0$. Then the waves' rays are the radial null geodesics $\chi = \eta - \eta_e$, where η_e is the coordinate time at which the ray is emitted. Retarded time on each ray is the proper time of emission, $\tau_e = \int_0^{\eta_e} a d\eta$; and the ray's radial function is $r = a \sin\chi$, which is the same radial function as appears in the expression for the optical brightness of a source of electromagnetic radiation [Eq. (29.28) of MTW].

Expressed in terms of the "deceleration parameter" q_o and "Hubble expansion rate" H_o of the universe, and the redshift z of the source as viewed optically from earth, this radial function is [Eq. (29.33) of MTW]

$$r = \frac{{H_o}^{-1}}{q_o^2(1+z)}[-q_o + 1 + q_o z + (q_o - 1)(2q_o + 1)^{1/2}] \; . \tag{5.4}$$

Note that, if the waves are emitted in an epoch when the universe's expansion factor is a_e and are received at earth when it is a_o, then in terms of proper time t as measured in the earth's local proper reference frame, the source's retarded time is $\tau_e = (a_e/a_o)t + \text{const.} = t/(1+z) + \text{const.}$, where $z = a_o/a_e$ is the source's "cosmological redshift". Correspondingly, the time dependence of the waveform as measured at earth is unchanged by the waves' propagation, except for a frequency-independent redshift $f_{\text{received}}/f_{\text{emitted}} = 1/(1+z)$ which is identically the same as for electromagnetic waves.

In fact, this similarity to electromagnetic waves is completely general: If one develops the geometric-optics formalism for electromagnetic wave propagation, one finds that the electromagnetic vector potential has the form

$$A_\alpha = A_1 a_\alpha + A_2 b_\alpha \; , \tag{5.5a}$$

where a_α and b_α are precisely the same polarization vectors as we used for gravitational waves; and that

$$A_1 = \frac{Q_1(\tau_e, \theta, \phi)}{r}, \qquad A_2 = \frac{Q_2(\tau_e, \theta, \phi)}{r} \; , \tag{5.5b}$$

where τ_e, θ, ϕ, r are precisely the same functions as we used above in the gravitational-wave formulas. (See, e.g., the treatment of monochromatic, electromagnetic geometric optics in Sec. 22.5 of MTW, translated into the notation of this book and with many frequencies superposed to give waveforms with arbitrary time dependences.) By comparing Eqs. (5.2e) and (5.5b) we infer that, in the geometric optics limit, the gravitational-wave fields h_+ and $h_\times$ experience precisely the same amplitude changes and redshift changes as do the components A_1 and A_2 of the electromagnetic vector potential.

5.B. Linear Perturbations of Curved Spacetime

As a foundation for deriving the above geometric-optics description of vacuum wave propagation (and also for discussing the effects discarded by geometric optics, the energy and momentum carried by gravitational waves, the generation of gravitational waves, and propagation through nonvacuum regions of spacetime), we shall develop in this section some mathematical formalism. This formalism will describe linearized perturbations of an arbitrary, nonvacuum spacetime geometry.

Whereas our description of gravitational waves thus far has focussed on the Riemann curvature tensor, our analysis here will focus on the metric $g_{\alpha\beta}$ of spacetime. We shall write that metric as the sum of a *background* metric $g^{\mathrm{B}}_{\alpha\beta}$ and a perturbation $h_{\alpha\beta}$:

$$g_{\alpha\beta} = g^{\mathrm{B}}_{\alpha\beta} + h_{\alpha\beta} \; . \tag{5.6a}$$

The precise way of making the split into background plus perturbation will depend on the situation. The background might be a smooth part, defined by averaging $g_{\alpha\beta}$ over several wavelengths of gravitational radiation, and the perturbation then will be the remaining, rippled part; or the background might be a spherically symmetric part, and the perturbation the deviations from spherical symmetry; or the background might be the equilibrium spacetime for a rotating black hole, and the perturbation the deviations from that equilibrium. In order to be able to handle, with one formalism, all these situations and more, we shall here regard the split (5.6a) as a purely formal one: The *background* metric $g^{\rm B}_{\alpha\beta}$ is one solution to the Einstein field equation with one stress-energy tensor $T^{\alpha\beta}_{\rm B}$ as its source; the *full* metric $g_{\alpha\beta}$ is another solution, with another stress-energy tensor $T^{\alpha\beta}$; and the two solutions are nearly but not quite the same. Nearly identical coordinate systems are set up in the spacetimes of these two solutions, the coordinates are given the same names x^μ, and events in the background spacetime and the full spacetime are regarded as "the same" if they have the same coordinate values. The perturbation $h_{\alpha\beta}$ at location x^μ is then the difference between the two functions $g_{\alpha\beta}(x^\mu)$ and $g^{\rm B}_{\alpha\beta}(x^\mu)$.

We shall regard the metric perturbation $h_{\alpha\beta}$ as a symmetric tensor field that lives in the background spacetime, and correspondingly, we shall raise and lower indices on $h_{\alpha\beta}$ using the background metric $g^{\rm B}_{\alpha\beta}$. Because $g^{\alpha\beta}$ is the inverse of $g_{\alpha\beta}$ ($g_{\alpha\beta}g^{\beta\gamma} = \delta_\alpha{}^\gamma$), $g^{\alpha\beta}$ takes the form

$$g^{\alpha\beta} = g^{\alpha\beta}_{\rm B} - h^{\alpha\beta} \; , \qquad \text{where } h^{\alpha\beta} = g^{\alpha\mu}_{\rm B} g^{\beta\nu}_{\rm B} h_{\mu\nu} \; . \tag{5.6b}$$

Note the opposite signs in expressions (5.6a,b) for $g_{\alpha\beta}$ and $g^{\alpha\beta}$, and note that there is no significance to whether the B (for background) is placed up or down.

The difference between the stress-energy tensor of the full spacetime and that of the background spacetime, at the same coordinate locations x^μ, we shall write in the form

$$T^{\alpha\beta} = T^{\alpha\beta}_{\rm B} + \mathcal{T}^{\alpha\beta} - h^{\mu(\alpha} T^{\beta)}_{\rm B}{}_\mu \; . \tag{5.7a}$$

Here and henceforth the parentheses on indices denote symmetrization:

$$h^{\mu(\alpha} T^{\beta)}_{\rm B}{}_\mu \equiv 1/2 (h^\mu \alpha T^\beta_{\rm B}{}_\mu + h^\mu \beta T^\alpha_{\rm B}{}_\mu) \; .$$

Equation (5.7a) serves as a definition of the stress-energy perturbation field $\mathcal{T}^{\alpha\beta}$. We choose this specific definition because it simplifies subsequent equations [e.g., Eqs. (5.33) and (5.34)]. We shall regard this $\mathcal{T}^{\alpha\beta}$, like $h_{\alpha\beta}$, as a linear field that resides in the background spacetime, with indices to be raised and lowered using $g^{\rm B}_{\alpha\beta}$. By lowering indices on $T^{\alpha\beta}$ with $g_{\alpha\beta} = g^{\rm B}_{\alpha\beta} + h_{\alpha\beta}$ and linearizing in $\mathcal{T}^{\alpha\beta}$ and $h_{\alpha\beta}$, we obtain

$$T_{\alpha\beta} = T^{\rm B}_{\alpha\beta} + \mathcal{T}^{\alpha\beta} + h^\mu{}_{(\alpha} T^{\rm B}_{\beta)\mu} \; . \tag{5.7b}$$

Note the opposite signs on the last terms in expressions (5.7a) and (5.7b), completely analogous to the opposite signs in (5.6a) and (5.6b).

The evolution of $\mathcal{T}^{\alpha\beta}$ will be governed by first-order perturbations of the law of energy-momentum conservation; and the evolution of $h_{\alpha\beta}$, by first-order perturbations of the

Einstein field equation. In discussing these evolution laws, we shall use a semicolon to denote a covariant derivative in the full spacetime, a vertical bar for a covariant derivative in the background spacetime, and a comma for an ordinary partial derivative with respect to our chosen coordinates. Correspondingly, the laws of energy-momentum conservation in the full and background spacetimes are

$$T^{\alpha\beta}{}_{;\beta} \equiv T^{\alpha\beta}{}_{,\beta} + \Gamma^{\alpha}{}_{\mu\beta}T^{\mu\beta} + \Gamma^{\beta}{}_{\mu\beta}T^{\alpha}\mu = 0 \; ; \tag{5.8a}$$

$$T^{\alpha\beta}_{\mathrm{B}}{}_{|\beta} \equiv T^{\alpha\beta}_{\mathrm{B}}{}_{,\beta} + \Gamma^{\mathrm{B}\,\alpha}{}_{\mu\beta}T^{\mu\beta}_{\mathrm{B}} + \Gamma^{\mathrm{B}\,\beta}{}_{\mu\beta}T^{\alpha\mu}_{\mathrm{B}} = 0 \; ; \tag{5.8b}$$

Here $\Gamma^{\alpha}{}_{\beta\gamma}$ is the connection coefficient for the full spacetime, and $\Gamma^{\mathrm{B}\,\alpha}{}_{\beta\gamma}$ is that for the background spacetime:

$$\Gamma^{\alpha}{}_{\beta\gamma} = 1/2 g^{\alpha\mu}(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu}) \; , \quad \Gamma^{\mathrm{B}\,\alpha}{}_{\beta\gamma} = 1/2 g^{\alpha\mu}_{\mathrm{B}}(g^{\mathrm{B}}_{\mu\beta,\gamma} + g^{\mathrm{B}}_{\mu\gamma,\beta} - g^{\mathrm{B}}_{\beta\gamma,\mu}) \; ; \tag{5.9}$$

[see, e.g., Eqs. (8.24b,c) of MTW]. Their difference can be evaluated to linear order with the help of Eqs. (5.6). The result, expressed in terms of the background's covariant derivative of $h_{\alpha\beta}$, is

$$S^{\alpha}{}_{\beta\gamma} \equiv \Gamma^{\alpha}{}_{\beta\gamma} - \Gamma^{\mathrm{B}\,\alpha}{}_{\beta\gamma} = 1/2 g^{\alpha\mu}_{\mathrm{B}}(h_{\mu\beta|\gamma} + h_{\mu\gamma|\beta} - h_{\beta\gamma|\alpha}) \; . \tag{5.10}$$

(For a sophisticated method of deriving this and the equations that follow, see Exercise 35.11 of MTW. However, sophistication is not needed; one can derive these equations by elementary algebra and index shuffling, plus a lot of sweat.)

It is a straightforward calculation to take the difference between the full and the background laws of energy-momentum conservation [Eqs. (5.8a,b)] and express it in terms of the perturbations of the stress-energy tensor and of the connection coefficients. The result, after also using Eq. (5.10), is the following *evolution equation for the stress-energy perturbation*:

$$\mathcal{T}^{\alpha\mu}{}_{|\mu} = 1/2(h_{\mu\nu}{}^{|\alpha} - h^{\alpha}{}_{\mu|\nu})T^{\mu\nu}_{\mathrm{B}} + 1/2(h_{\mu\nu}{}^{|\nu} - h^{\nu}{}_{\nu|\mu})T^{\mu\alpha}_{\mathrm{B}} \; . \tag{5.11}$$

This shows how gradients of the metric perturbations $h_{\alpha\beta|\gamma}$ couple to the background stress-energy to generate stress-energy perturbations $\mathcal{T}_{\alpha\beta}$.

One application of this equation is to gravitational-wave detection: there $h_{\alpha\beta}$ is the metric perturbation associated with the gravitational radiation, $T^{\alpha\beta}_{\mathrm{B}}$ is the stress-energy tensor of an unperturbed detector, and $\mathcal{T}^{\alpha\beta}$ is the influence of the waves on the detector. When the detector involves a set of masses on which the waves act, and the coordinate system is a proper reference frame of $g_{\alpha\beta}$ and also of $g^{\mathrm{B}}_{\alpha\beta}$, in which the masses are nearly at rest, then the dominant spatial term (driving force) on the right-hand side of (5.11) is

$$1/2 h_{00}{}^{|j}T^{00}_{\mathrm{B}} = -(R^{\mathrm{GW}}_{j0k0}x^k)T^{00}_{\mathrm{B}}$$

[cf. Eq. (4.39)]. This is the "per-unit-volume" version of the gravitational-wave force (4.12) of Chap. 4.

The Riemann curvature tensors of our two spacetimes can be expressed in terms of the connection coefficients and their derivatives by the standard formula [MTW, Eq. (8.44)]

$$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha{}_{\beta\delta,\gamma} - \Gamma^\alpha{}_{\beta\gamma,\delta} + \Gamma^\alpha{}_{\mu\gamma}\Gamma^\mu{}_{\beta\delta} - \Gamma^\alpha{}_{\mu\delta}\Gamma^\mu{}_{\beta\gamma} \ , \tag{5.12}$$

and the same formula for the background but with a superscript B on all quantitities. By taking the difference between these two formulas and discarding terms nonlinear in the metric perturbation, we obtain

$$\begin{aligned} \delta R^\alpha{}_{\beta\gamma\delta} &\equiv R^\alpha{}_{\beta\gamma\delta} - R^{\mathrm{B}\,\alpha}{}_{\beta\gamma\delta} = S^\alpha{}_{\beta\delta|\gamma} - S^\alpha{}_{\beta\gamma|\delta} \\ &= 1/2 g_{\mathrm{B}}^{\alpha\mu}(h_{\mu\beta|\delta\gamma} + h_{\mu\delta|\beta\gamma} - h_{\beta\delta|\mu\gamma} - h_{\mu\beta|\gamma\delta} - h_{\mu\gamma|\beta\delta} + h_{\beta\gamma|\mu\delta}) \ . \end{aligned} \tag{5.13}$$

Since the Ricci curvature tensors are obtained by contracting on the first and third indices of the Riemann tensors, Eq. (5.13) yields immediately for the perturbation of the Ricci tensor

$$\delta R_{\beta\delta} \equiv R_{\beta\delta} - R^{\mathrm{B}}_{\beta\delta} = 1/2(h_{\mu\beta|\delta}{}^\mu + h_{\mu\delta|\beta}{}^\mu - h_{\beta\delta|\mu}{}^\mu - h_{|\beta\delta}) \ , \tag{5.14}$$

where the h with no subscripts denotes the contraction of $h_{\mu\nu}$ (using, of course, the background metric: $h \equiv h_{\mu\nu} g_{\mathrm{B}}^{\mu\nu}$).

It is the Einstein tensor, $G_{\mu\nu} \equiv R_{\mu\nu} - 1/2 R g_{\mu\nu}$ (where $R \equiv R_{\alpha\beta} g^{\alpha\beta}$) that appears on the left-hand side of the Einstein field equation. The perturbation in the Einstein tensor is readily computed from Eq. (5.14) and the perturbation (5.6) in the metric. The result is most nicely expressed not in terms of the metric perturbation $h_{\alpha\beta}$, but rather in terms of the trace-reversed metric perturbation

$$\bar{h}_{\alpha\beta} \equiv h_{\alpha\beta} - 1/2 h g^{\mathrm{B}}_{\alpha\beta} \ . \tag{5.15}$$

The result is brought into the nicest form by commuting a pair of background covariant derivatives [at the price of introducing a term proportional to the background Riemann tensor; cf. Eqs. (16.6) of MTW]. The resulting, "nicest form" is

$$\begin{aligned} G_{\alpha\beta} - G^{\mathrm{B}}_{\alpha\beta} \equiv G^{(1)}_{\alpha\beta}(h) = &- 1/2(\bar{h}^\mu_{\alpha\beta|\mu} + g^{\mathrm{B}}_{\alpha\beta}\bar{h}^{\mu\nu}{}_{|\mu\nu} - 2\bar{h}_{\mu(\alpha|}{}^\mu{}_{\beta)} + 2R^{\mathrm{B}}_{\mu\alpha\nu\beta}\bar{h}^{\mu\nu} \\ &- 2R^{\mathrm{B}}_{\mu(\alpha}\bar{h}_{\beta)}{}^\mu - R^{\mathrm{B}}_{\mu\nu}\bar{h}^{\mu\nu} g^{\mathrm{B}}_{\alpha\beta} + R^{\mathrm{B}}\bar{h}_{\alpha\beta}) \ . \end{aligned} \tag{5.16}$$

Here $R^{\mathrm{B}} \equiv R^{\mathrm{B}\mu}{}_\mu$ is the background scalar curvature, and the parentheses on indices denote symmetrization. This is the nicest form of $G^{(1)}_{\alpha\beta}$ because it simplifies so much when one specializes the *gauge*:

By "gauge" we mean "the choice of which points in the full spacetime correspond to which points in the background spacetime". To change the gauge, we can hold fixed the coordinates in the background spacetime, but change those in the full spacetime by the following very small amount (i.e., the following "infinitesimal coordinate transformation"):

$$x^\alpha_{\mathrm{old}} = x^\alpha_{\mathrm{new}} - \xi^\alpha(x^\mu_{\mathrm{new}}) \ . \tag{5.17}$$

This gauge change produces the following change in the metric coefficients of the full spacetime:

$$g^{\rm new}_{\alpha\beta}(x^\mu_{\rm new}) = g^{\rm old}_{\rho\sigma}(x^\mu_{\rm old}) \frac{\partial x^\sigma_{\rm old}}{\partial x^\alpha_{\rm new}} \frac{\partial x^\sigma_{\rm old}}{\partial x^\beta_{\rm new}} \; . \tag{5.18}$$

By combining this with Eq. (5.17) and with Eq. (5.6) for both the old and the new gauges, we obtain the relationship between the new and the old metric perturbations

$$h^{\rm new}_{\alpha\beta} = h^{\rm old}_{\alpha\beta} - \xi_{\alpha|\beta} - \xi_{\beta|\alpha} \; . \tag{5.19}$$

Thus, the *generator of the gauge change* ξ_α, viewed as a field living in the background spacetime, produces the change (5.19) in the metric perturbation. This is completely analogous to gauge changes in electromagnetism: The $h_{\alpha\beta}$ here is the analog of the electromagnetic vector potential A_α; the generator of the gauge change ξ_α is the analog of the electromagnetic gauge-change generator ψ; and Eq. (5.19) is the analog of $A^{\rm new}_\alpha = A^{\rm old}_\alpha - \psi_{,\alpha}$. Moreover, just as the electromagnetic evolution equation for A_α takes on an especially simple form when one specializes to the "Lorentz gauge" ($A^\alpha{}_{|\alpha} = 0$), so also the Einstein-tensor perturbation and thence the Einstein equation take on especially simple forms when one specializes to the *gravitational Lorentz gauge*

$$\bar h^{\alpha\beta}_{|\beta} = 0 \; . \tag{5.20}$$

If one is not initially in this Lorentz gauge, one can go there by the gauge change whose generator satisfies the wave-equation-with-source

$$\xi_{\alpha|\beta}{}^\beta = -R^{\rm B}_{\alpha\beta}\xi^\beta + h^{\rm old}_{\alpha\beta}\,|\beta \; . \tag{5.21}$$

In Lorentz gauge the perturbation (5.16) of the Einstein tensor simplifies to

$$G^{(1)}_{\alpha\beta}(h) = -1/2(\bar h_{\alpha\beta|\mu}{}^\mu + 2R^{\rm B}_{\mu\alpha\nu\beta}\bar h^{\mu\nu} - 2R^{\rm B}_{\mu(\alpha}\bar h_{\beta)}{}^\mu - R^{\rm B}_{\mu\nu}\bar h^{\mu\nu} g^{\rm B}_{\alpha\beta} + R_{\rm B}\bar h_{\alpha\beta}) \; , \tag{5.22}$$

which involves only the wave operator plus the coupling of $\bar h_{\alpha\beta}$ to background curvature.

The perturbed Einstein equation [with Newton's gravitation constant G set to unity as we shall do throughout Chaps. 5 and 6, cf. Eqs. (4.3)] equates this $G^{(1)}_{\alpha\beta}$ to 8π times the stress-energy perturbation [Eq. (5.7b)]

$$G^{(1)}_{\alpha\beta}(h) = 8\pi(T_{\alpha\beta} - T^{\rm B}_{\alpha\beta}) = 8\pi(\mathcal{T}_{\alpha\beta} + h^\mu{}_{(\alpha}T^{\rm B}_{\beta)\mu}) \; . \tag{5.23}$$

By combining Eqs. (5.22), (5.23), and the background Einstein equation $G^{\alpha\beta}_{\rm B} = 8\pi T^{\alpha\beta}_{\rm B}$, we obtain our final Lorentz-gauge form for the first-order, perturbed Einstein equation:

$$\bar h_{\alpha\beta|\mu}{}^\mu + 2R^{\rm B}_{\mu\alpha\nu\beta}\bar h^{\mu\nu} = -16\pi(\mathcal{T}_{\alpha\beta} - 1/2\bar h_{\mu\nu}T^{\mu\nu}_{\rm B} g^{\rm B}_{\alpha\beta} - 1/2\bar h T^{\rm B}_{\alpha\beta} + 1/4\bar h T_{\rm B} g^{\rm B}_{\alpha\beta}) \; . \tag{5.24}$$

The equation of motion (5.11) for $\mathcal{T}_{\alpha\beta}$, which goes hand-in-hand with this wave equation, takes the following form when expressed in terms of $\bar h_{\alpha\beta}$ rather than $h_{\alpha\beta}$, and when specialized to Lorentz gauge:

$$\mathcal{T}^{\alpha\mu}{}_{|\mu} = 1/2(\bar h_{\mu\nu}{}^{|\alpha} - \bar h^\alpha{}_{\mu|\nu})T^{\mu\nu}_{\rm B} + 1/2\bar h_{|\mu}T^{\mu\alpha}_{\rm B} - 1/4\bar h^{|\alpha}T_{\rm B} \; . \tag{5.25}$$

In Eqs. (5.24) and (5.25), $\bar{h}$ and $T_{\rm B}$ without indices denote the traces of $\bar{h}_{\alpha\beta}$ and $T_{\rm B}^{\alpha\beta}$: $\bar{h} \equiv \bar{h}_{\alpha\beta} g_{\rm B}^{\alpha\beta}$ and $T_{\rm B} \equiv T_{\rm B}^{\alpha\beta} g^{\rm B}_{\alpha\beta}$.

Equations (5.24) and (5.25) describe the joint, coupled evolution of the gravitational perturbations $\bar{h}_{\alpha\beta}$ *and the stress-energy perturbations* $\mathcal{T}_{\alpha\beta}$. In the remainder of this chapter we shall use these coupled equations to study the propagation of gravitational waves in the presence of matter and electromagnetic fields. In Chap. 6 we shall use them to study the generation of gravitational waves by astrophysical sources.

5.C. Shortwave Formalism

Turn, now, from general formalism to a specific situation: the propagation of gravitational radiation with reduced wavelength $\bar{\lambda}$ through a background spacetime with inhomogeneity scale $\mathcal{L}$. Assume, as in Chap. 4, that $\bar{\lambda} \ll \mathcal{L}$ so the waves are well defined. Then the propagation is beautifully described using a *shortwave formalism* due to Isaacson (1968a,b). In this section we shall develop that formalism.

In our analysis we shall need, in addition to $\bar{\lambda}$ and $\mathcal{L}$, a third lengthscale: the radius of curvature $\mathcal{R}$ of the background spacetime. We define it to be the inverse square root of the largest components of the background Riemann tensor, evaluated in a proper reference frame

$$\mathcal{R} \equiv (\text{minimum value of } |R^{\rm B}_{\alpha\beta\gamma\delta}|^{-1/2}) \;: \tag{5.26}$$

This lengthscale is depicted heuristically in Fig. 4.1 on page XX. We shall always define the inhomogeneity lengthscale $\mathcal{L}$ to be no larger than the curvature lengthscale,

$$\mathcal{L} \lesssim \mathcal{R} \;, \tag{5.27}$$

because of a philosophical viewpoint that curvature itself is a form of inhomogeneity. (This also simplifies some of the conceptual issues that follow.)

We introduce into the full spacetime a coordinate system which, for the moment, we constrain in only one way: we demand that in it the metric coefficients $g_{\alpha\beta}$, like the physical curvature, vary on lengthscales $\bar{\lambda}$, $\sim \mathcal{L}$, and possibly $\gg \mathcal{L}$, but not on any lengthscales between $\bar{\lambda}$ and $\mathcal{L}$.

Following Isaacson (1968a), we shall call such a coordinate system *steady*. We then define the background spacetime by the demand that there exist in it coordinates in which the background metric coefficients are the average over several wavelengths of the steady coordinates' $g_{\alpha\beta}$:

$$g^{\rm B}_{\alpha\beta}(x^\mu) \equiv \left\langle g_{\alpha\beta}(x^\mu) \right\rangle \,. \tag{5.28}$$

The difference between $g_{\alpha\beta}$ and $g^{\rm B}_{\alpha\beta}$ describes, of course, the gravitational radiation. In analyzing this radiation we shall be interested not only in linear effects, but also in nonlinear ones—for example, the energy and momentum carried by the waves. In preparation for discussing nonlinear effects, we shall write the difference between $g_{\alpha\beta}$ and $g^{\rm B}_{\alpha\beta}$ not as a single field $h_{\alpha\beta}$, but rather as a power series:

$$g_{\alpha\beta} = \underset{1,\mathcal{L}}{g^{\rm B}_{\alpha\beta}} + \underset{h\bar{\lambda}}{h_{\alpha\beta}} + \underset{h^2\bar{\lambda}}{j_{\alpha\beta}} + \;\ldots\;. \tag{5.29}$$

Below each term we show the characteristic magnitude $(1, h, h^2)$ of the term, and also the lengthscale $\bar\lambda$ or $\mathcal{L}$) on which it varies. Note that $j_{\alpha\beta}$ is a nonlinear correction to the propagating waves. This nonlinear correction is not yet precisely defined. We are free to shove pieces of it in and out of $h_{\alpha\beta}$ in such a way as to make the computational formalism as simple as possible.

Similarly, we split up the covariant components of the stress-energy tensor in the following way:

$$T_{\alpha\beta} = T^{\rm B}_{\alpha\beta} + \mathcal{T}_{\alpha\beta} + h^{\mu}{}_{(\alpha}T^{\rm B}_{\beta)\mu} + j^{\mu}{}_{(\alpha}T^{\rm B}_{\beta)\mu} + \ldots \tag{5.30a}$$

[cf. Eq. (5.7b)]. Here, by definition, $T^{\rm B}_{\alpha\beta}$ is the average of $T_{\alpha\beta}$ over several wavelengths

$$T^{\rm B}_{\alpha\beta} \equiv \langle T_{\alpha\beta} \rangle \; ; \tag{5.30b}$$

and $\mathcal{T}_{\alpha\beta} + h_{\mu(\alpha}T^{\rm B}_{\beta)\mu} + j^{\mu}{}_{(\alpha}T^{\rm B}_{\beta)\mu} + \ldots$ is the fluctuating part, which averages to zero. Our chosen form (5.30a) of $T_{\alpha\beta}$ amounts to a definition of the stress-energy perturbation field $\mathcal{T}_{\alpha\beta}$. This specific definition is carefully chosen to make the fluctuating parts of the field equations [Eqs. (5.33) and (5.34) below] especially simple. We shall meet evidence of that simplicity in specific calculations with the formalism in Sec. 5.E.

In formulating the mathematics of our shortwave formalism, as in the linear perturbation theory of Sec. 5.C, we shall treat $h_{\alpha\beta}$, $j_{\alpha\beta}$, $\mathcal{T}_{\alpha\beta}$, and all other quantities except $T_{\alpha\beta}$ and $g_{\alpha\beta}$, as fields that reside in the background spacetime. Correspondingly, we shall raise and lower their indices using the background metric $g^{\rm B}_{\alpha\beta}$.

By a calculation analogous to that of the last section, but one which includes nonlinear terms as well as linear, one can derive a power-series expansion for the Einstein curvature tensor $G_{\alpha\beta}$ of the full spacetime:

$$G_{\alpha\beta} = \underset{\lesssim \mathcal{R}^{-2},\, \mathcal{L}}{G^{\rm B}_{\alpha\beta}} + \underset{h\bar\lambda^{-2},\, \bar\lambda}{G^{(1)}_{\alpha\beta}(h)} + \underset{h^2\bar\lambda^{-2},\, \bar\lambda}{G^{(2)}_{\alpha\beta}(h)} + \underset{h^2\bar\lambda^{-2},\, \bar\lambda}{G^{(1)}_{\alpha\beta}(j)} + \ldots\, . \tag{5.31}$$

Here, as in Eq. (5.29), we write below each term its magnitude and the lengthscale on which it varies. The notation has the following meanings: $G^{\rm B}_{\alpha\beta}$ is the Einstein curvature tensor of the background spacetime, computed from the metric $g^{\rm B}_{\alpha\beta}$; $G^{(1)}_{\alpha\beta}(h)$, the piece that is linear in the radiation field $h_{\alpha\beta}$, is given by expression (5.16); $G^{(1)}_{\alpha\beta}(j)$ is this same expression, but with $h_{\alpha\beta}$ replaced by $j_{\alpha\beta}$; and $G^{(2)}_{\alpha\beta}(h)$ is the piece that is quadratic in $h_{\alpha\beta}$ [derivable from MTW Eqs. (35.58)].

Following Isaacson (1968a,b), we split the Einstein field equation $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$, through order h^2, into three parts: a part which varies on scales $\mathcal{L}$ (obtained by averaging over a few wavelengths)

$$G^{\rm B}_{\alpha\beta} = -\big\langle G^{(2)}_{\alpha\beta}(h)\big\rangle + 8\pi T^{\rm B}_{\alpha\beta} \; ; \tag{5.32a}$$

a part whose individual terms have magnitude $h\bar\lambda^{-2}$ or smaller, vary on scales $\bar\lambda$, and average to zero on larger scales

$$G^{(1)}_{\alpha\beta}(h) = 8\pi(\mathcal{T}_{\alpha\beta} + h^{\mu}{}_{(\alpha}T^{\rm B}_{\beta)\mu}) \; ; \tag{5.32b}$$

and a part whose terms have magnitude $h\lambda\!\!\!^{-}{}^{-2}$ or smaller, vary on scales $\lambda\!\!\!^{-}$, and average to zero on larger scales

$$G^{(1)}_{\alpha\beta}(j) = -G^{(2)}_{\alpha\beta}(h) + \left\langle G^{(2)}_{\alpha\beta}(h)\right\rangle + 8\pi j^{\mu}{}_{(\alpha}T^{\rm B}_{\beta)\mu} \; . \tag{5.32c}$$

This choice of how to split up the field equations determines the details of the split of the waves into $h_{\alpha\beta}+j_{\alpha\beta}$. Changing the split-up [pulling a piece of Eq. (5.32c) into Eq. (5.32b)] would shove a piece of $j_{\alpha\beta}$ into $h_{\alpha\beta}$. Our specific choice of the split is guided by a desire to make the first-order equation (5.32b) identical to the linearized equation (5.23), so that when a weak wave enters a region of rapidly varying curvature $\mathcal{L} \lesssim \lambda\!\!\!^{-}$ (e.g., when it impinges on a black hole), our first-order equation continues to be valid.

The first-order equation (5.32b) is a wave equation for the propagation of the first-order gravitational wave $h_{\alpha\beta}$. By specializing to Lorentz gauge, expressing the background Ricci tensor in terms of the background Einstein tensor, using Eq. (5.36) below for the background Einstein tensor, and discarding terms of the form $\bar{h}_{\alpha\beta}T^{\rm GW}_{\gamma\delta}$ because they are cubic in the wave amplitude h (one order smaller than the accuracy of our analysis), we bring Eq. (5.32b) into the standard linearized form (5.24):

$$\Box\bar{h}_{\alpha\beta} \equiv \bar{h}_{\alpha\beta|\mu}{}^{\mu} = -2R^{\rm B}_{\mu\alpha\nu\beta}\bar{h}^{\mu\nu} - 16\pi(\mathcal{T}_{\alpha\beta} - 1/2\bar{h}_{\mu\nu}T^{\mu\nu}_{\rm B}g^{\rm B}_{\alpha\beta} - 1/2\bar{h}T^{\rm B}_{\alpha\beta} + 1/4\bar{h}T_{\rm B}g^{\rm B}_{\alpha\beta}) \; . \tag{5.33}$$

Much of the rest of this chapter will be devoted to a discussion of this wave equation and the physical effects associated with it.

In Lorentz gauge and in vacuum, Eq. (5.32c) takes on the form

$$\begin{aligned}\Box\bar{\jmath}_{\alpha\beta} =& -2R^{\rm B}_{\mu\alpha\nu\beta}\bar{\jmath}^{\mu\nu} + 8\pi(\bar{\jmath}_{\mu\nu}T^{\mu\nu}_{\rm B}g^{\rm B}_{\alpha\beta} + \bar{\jmath}T^{\rm B}_{\alpha\beta} - 1/2\bar{\jmath}T_{\rm B}g^{\rm B}_{\alpha\beta}) \\ &+ 2G^{(2)}_{\alpha\beta}(h) - 2\langle G^{(2)}_{\alpha\beta}(h)\rangle \; .\end{aligned} \tag{5.34}$$

Here $\bar{\jmath}_{\alpha\beta} \equiv j_{\alpha\beta} - 1/2 j_{\mu}{}^{\mu}g^{\rm B}_{\alpha\beta}$, and $\bar{\jmath} \equiv \bar{\jmath}_{\mu}{}^{\mu}$. The terms on the right-hand side involving $G^{(2)}(h)$, which are quadratic in $h_{\alpha\beta}$, act as a source for the nonlinear corrections $j_{\alpha\beta}$ to $h_{\alpha\beta}$. Thus, this equation describes nonlinear wave-wave coupling ("3-wave coupling" in the standard jargon of nonlinear physics) analogous to that which occurs in plasma physics or for electromagnetic waves in a nonlinear medium. We shall discuss the effects of this wave-wave coupling in Sec. 5.G.e below.

Equation (5.32a) describes the waves' nonlinear generation of background curvature. This equation, in fact, is the foundation for Isaacson's (1968b) description of the energy and momentum carried by the waves: Isaacson defines the *gravitational-wave stress-energy tensor* by

$$T^{\rm GW}_{\alpha\beta} \equiv -\frac{1}{8\pi}\left\langle G^{(2)}_{\alpha\beta}(h)\right\rangle \; , \tag{5.35}$$

and then notes that in terms of this tensor Eq. (5.32a) takes on the same form as the standard Einstein field equation

$$G^{\rm B}_{\alpha\beta} = 8\pi(T^{\rm B}_{\alpha\beta} + T^{\rm GW}_{\alpha\beta}) \; . \tag{5.36}$$

Equation (5.36) shows that the gravitational-wave stress-energy tensor, like any nongravitational stress-energy tensor, generates spacetime curvature. Isaacson points out, moreover, that because $G^{\rm B}_{\alpha\beta}$, like any Einstein tensor, automatically has vanishing divergence, the sum $T^{\rm B}_{\alpha\beta} + T^{\rm GW}_{\alpha\beta}$ is guaranteed also to have vanishing divergence:

$$(T^{{\rm B}\,\alpha\beta} + T^{{\rm GW}\,\alpha\beta})_{|\beta} = 0 \; . \tag{5.37}$$

In other words, when averaged over a few wavelengths, the sum of the gravitational-wave energy-momentum and the nongravitational energy-momentum is conserved. For example, when a gravitational-wave detector is driven into motion by a passing wave, the detector's energy (embodied in $T^{{\rm B}\,\alpha\beta}$) goes up, and the wave's energy (embodied in $T^{{\rm GW}\,\alpha\beta}$) goes down.

A straightforward but tedious calculation (Isaacson, 1968b; Eq. (35.70) of MTW and associated discussion) gives the following explicit expression for $T^{\rm GW}_{\alpha\beta}$ in an arbitrary gauge:

$$T^{\rm GW}_{\alpha\beta} = \frac{1}{32\pi}\left\langle \bar{h}_{\mu\nu|\alpha}\bar{h}^{\mu\nu}_{|\beta} - 1/2\bar{h}_{|\alpha}\bar{h}_{|\beta} - \bar{h}^{\mu\nu}{}_{|\nu}\bar{h}_{\mu\alpha|\beta} - \bar{h}^{\mu\nu}{}_{|\nu}\bar{h}_{\mu\beta|\alpha}\right\rangle \; . \tag{5.38}$$

Here $\bar{h}_{\alpha\beta}$ is the trace-reversed, first-order metric perturbation [Eq. (5.15)]. Note that in Lorentz gauge the last two terms vanish; and in a nearly Lorentz frame and TT gauge, because $\bar{h}_{00} = \bar{h}_{0j} = 0$ and $\bar{h}_{jk} = h^{\rm GW}_{jk}$ which is trace free, Eq. (5.38) reduces to

$$T^{\rm GW}_{\alpha\beta} = \frac{1}{32\pi}\left\langle h^{\rm GW}_{jk,\alpha}h^{\rm GW}_{jk,\beta}\right\rangle \; . \tag{5.39}$$

Here there is an implied summation on j and k, which are Cartesian, spatial indices. When, moreover, the waves propagate in the z-direction of a nearly Lorentz frame so $h^{\rm GW}_{xx} = -h^{\rm GW}_{yy} = h_+(t-z)$, $h^{\rm GW}_{xy} = h^{\rm GW}_{yx} = h_\times(t-z)$, this becomes the expression (4.33) that was discussed in Chap. 4.

Note that the magnitude of the gravitational-wave stress-energy tensor is $T^{\rm GW}_{\alpha\beta} \sim (h/\lambda\!\!\!^{-})^2$; cf. Eq. (4.34). Since this stress-energy is a source of background curvature through the averaged Einstein equation $G^{\rm B}_{\alpha\beta} = 8\pi(T^{\rm B}_{\alpha\beta} + T^{\rm GW}_{\alpha\beta})$ [Eq. (5.36)], it must be that $G^{\rm B}_{\alpha\beta} \gtrsim (h/\lambda\!\!\!^{-})^2$. However, because $G^{\rm B}_{\alpha\beta}$ is constructed as a sum of components of the background Riemann tensor, the largest of which have magnitudes $1/\mathcal{R}^2$, it must be that $G^{\rm B}_{\alpha\beta} \lesssim 1/\mathcal{R}^2$. From these relations and $\mathcal{R} \gtrsim \mathcal{L}$ we infer that

$$h \lesssim \lambda\!\!\!^{-}/\mathcal{R} \lesssim \lambda\!\!\!^{-}/\mathcal{L} \; . \tag{5.40}$$

Since the very concept of a gravitational wave has meaning only when $\lambda\!\!\!^{-} \ll \mathcal{L}$, Eq. (5.40) tells us that *gravitational radiation always has a small dimensionless amplitude*,

$$h \ll 1 \; . \tag{5.41}$$

Equation (5.37) is only one portion of the law of conservation of energy-momentum: the portion obtained by averaging $T^{\alpha\beta}{}_{;\beta} = 0$ over a few wavelengths. The other portion, that which fluctuates on scales of order $\lambda\!\!\!^{-}$ and averages to zero, has the linearized form (5.25):

$$\mathcal{T}^{\alpha\mu}{}_{|\mu} = 1/2(\bar{h}_{\mu\nu}{}^{|\alpha} - \bar{h}^{\alpha}{}_{\mu|\nu})T^{\mu\nu}_{\rm B} + 1/2\bar{h}_{|\mu}T^{\mu\alpha}_{\rm B} - 1/4\bar{h}^{|\alpha}T_{\rm B} \; . \tag{5.42}$$

The right side is the force exerted by the waves on the matter or fields through which they propagate, and the left side is the response of the matter or fields to this gravitational force.

5.D. Geometric Optics

Turn, next, to the task of solving the wave equation (5.33) for the propagation of $\bar{h}^{\alpha\beta}$ from its source to the earth. As we shall see in Secs. 5.E and 5.F, in the real astrophysical universe (except near the Planck time) the effects of matter and electromagnetic fields on the propagating waves are minuscule. This permits us to simplify our analysis by setting to zero, in the propagation equation (5.33), the nongravitational stress-energies $\mathcal{T}_{\alpha\beta}$ and $T^{\mathrm{B}}_{\alpha\beta}$—a procedure we shall call the *vacuum approximation* for wave propagation.

The wave equation (5.33) has already been simplified by specializing to Lorentz gauge; and we shall now simplify it further by an additional specialization of the gauge: It is not difficult to verify that the gauge will remain Lorentz ($\bar{h}^{\alpha\beta}_{\ \ |\beta} = 0$ will continue to be satisfied) if the generator ξ_α of the gauge change (5.19) satisfies the wave equation $\Box\xi_\alpha = 0$. Note, further, that the trace of the propagation equation (5.33) guarantees (in vacuum) that $\bar{h} = \bar{h}^{\alpha}{}_{\alpha}$ satisfies the wave equation $\Box\bar{h} = 0$. Accordingly, if we choose ξ_α to be a solution of the wave equation $\Box\xi_\alpha = 0$ such that $\xi^{\alpha}{}_{|\alpha} = 1/2 h^{\mathrm{old}} = -1/2\bar{h}^{\mathrm{old}}$, then the gauge change (5.19) will remove the trace of $h_{\alpha\beta}$. We make this gauge change, thereby guaranteeing that

$$h = \bar{h} = 0 \,, \qquad \bar{h}_{\alpha\beta} = h_{\alpha\beta} \,. \tag{5.43}$$

This permits us to omit bars from $h_{\alpha\beta}$ in what follows.

We expect the field $h_{\alpha\beta}$ to be a rapidly varying function of the source's retarded time τ_e, and a slowly varying function of all other spacetime coordinates. More specifically, it should vary in τ_e on a lengthscale $\bar{\lambda}$ (the reduced wavelength of the waves); and all its other variations should have lengthscales no longer than

$$\mathcal{D} \equiv \text{minimum of } \mathcal{L} \text{ and radius of curvature of wave fronts.} \tag{5.44}$$

Accordingly, we shall seek a solution of the propagation equation which is accurate only to leading order in the small dimensionless parameter $\bar{\lambda}/\mathcal{D}$—in other words, we shall solve for the propagation using the *vacuum, geometric-optics approximation*.

As a formal mathematical tool in the solution, we introduce a parameter ϵ which tells us at a glance the relative orders of magnitude of various terms: If a specific term is of order $(\bar{\lambda}/\mathcal{D})^n$ relative to other terms with which it is compared, then we shall prepend to it a factor ϵ^n. However, we shall take the numerical value of ϵ to be one, thereby allowing ourselves to drop it when it ceases to be useful. In this spirit, we write the field $h_{\alpha\beta}$ in the form ("geometric optics expansion")

$$h_{\alpha\beta} = h^{[0]}_{\alpha\beta}(\tau_e/\epsilon, x^\mu) + \epsilon h^{[1]}_{\alpha\beta}(\tau_e/\epsilon,\, x^\mu) + \epsilon^2 h^{[2]}_{\alpha\beta}(\tau_e/\epsilon,\, x^\mu) + \ldots \,. \tag{5.45}$$

The term $h^{[0]}_{\alpha\beta}$ is the geometric optics approximation to $h_{\alpha\beta}$, and $h^{[1]}_{\alpha\beta}$, $h^{[2]}_{\alpha\beta}$, … are "post-geometric-optics corrections". It will be straightforward to read off our formalism the equations governing the corrections, but we will focus attention in the end only on the leading term, $h^{[0]}_{\alpha\beta}$. Each $h^{[n]}_{\alpha\beta}$ varies in τ_e on the scale $\bar{\lambda}$ and in its argument x^μ on the scale $\mathcal{D}$; i.e., it can be thought of as varying on scales of order unity in both $\tau_e/\bar{\lambda}$ and $x^\mu/\mathcal{D}$—which means that by comparison with x^μ, the τ_e in the functional form requires a factor

ϵ^{-1}. That is why we write the functional form as $h^{[n]}_{\alpha\beta}(\tau_e/\epsilon,\, x^\mu)$. Because of this functional form, we write the covariant derivative (gradient) of $h^{[n]}_{\alpha\beta}$ as

$$h^{[n]}_{\alpha\beta|\mu} = -\epsilon^{-1}\dot{h}^{[n]}_{\alpha\beta}k_\mu + h^{[n]}_{\alpha\beta|\mu'}\ . \tag{5.46}$$

Here the dot denotes a derivative with respect to τ_e

$$\dot{h}_{\alpha\beta} \equiv \left(\frac{\partial h_{\alpha\beta}}{\partial \tau_e}\right)_{x^\mu}\ ; \tag{5.47a}$$

k_μ is the negative of the gradient of τ_e

$$k_\mu \equiv -\tau_{e|\mu}\ ; \tag{5.47b}$$

and the prime on the last μ index indicates that the covariant derivative is to be taken holding τ_e constant.

This notation makes straightforward the geometric-optics expansion of the propagation equation (5.33), the Lorentz gauge condition (5.20), and our auxiliary gauge condition (5.43). The result of those expansions is:

$$\begin{aligned}
&\epsilon^{-2}\ddot{h}^{[0]}_{\alpha\beta}k_\mu k^\mu + \epsilon^{-1}(-2\dot{h}^{[0]}_{\alpha\beta|\mu'}k^\mu - \dot{h}^{[0]}_{\alpha\beta}{k^\mu}_{|\mu} + \ddot{h}^{[1]}_{\alpha\beta}k_\mu k^\mu) + \epsilon^0(2R^{\rm B}_{\mu\alpha\nu\beta}h^{[0]\,\mu\nu} \\
&\qquad + {h^{[0]}_{\alpha\beta|\mu'}}^{|\mu'} - 2\dot{h}^{[1]}_{\alpha\beta|\mu'}k^\mu - \dot{h}^{[1]}_{\alpha\beta}{k^\mu}_{|\mu} + \ddot{h}^{\alpha\beta[2]}k_\mu k^\mu) + {\rm O}(\epsilon) = 0\ ; && (5.48a)\\
&\qquad - \epsilon^{-1}\dot{h}^{[0]}_{\alpha\beta}k^\beta + \epsilon^0({h^{[0]}_{\alpha\beta}}^{|\beta'} - \dot{h}^{[1]}_{\alpha\beta}k^\beta) + {\rm O}(\epsilon) = 0\ ; && (5.48b)\\
&{h^{[0]}_\alpha}^\alpha + \epsilon {h^{[1]}_\alpha}^\alpha + \epsilon^2 {h^{[2]}_\alpha}^\alpha + {\rm O}(\epsilon^3) = 0\ . && (5.48c)
\end{aligned}$$

By equating to zero the leading two orders in (5.48a) and the leading order in (5.43b,c), we obtain the geometric-optics equations that govern $h^{[0]}_{\alpha\beta}$:

$$k_\mu k^\mu = 0\ , \quad h^{[0]}_{\alpha\beta|\mu'}k^\mu = -1/2{k^\mu}_{|\mu}h^{[0]}_{\alpha\beta}\ , \quad h^{[0]}_{\alpha\beta}k^\beta = 0\ , \quad {h^{[0]}_\alpha}^\alpha = 0\ . \tag{5.49}$$

The higher-order terms in (5.48) govern the post-geometric-optics corrections to the propagation.

The first of Eqs. (5.49) tells us that the wave vector $\mathbf{k} = -\nabla\tau_e$ is null, as it surely should be since we chose τ_e to be retarded time. By taking the gradient of $k_\mu k^\mu = 0$, then noting that $k_{\mu|\nu} = -\tau_{e_{|\mu\nu}} = -\tau_{e_{|\nu\mu}} = k_{\nu|\mu}$, we deduce that ${k_{\nu|\mu}}^{|\mu} = 0$. Thus, the wave vector is the tangent vector to a null geodesic. That null geodesic is a *ray* of the propagating waves.

The second of Eqs. (5.49) tells us how the field $h^{[0]}_{\alpha\beta}$ propagates along its rays. Note that, because τ_e is constant along any ray, we do not need the prime on the gradient in this propagation equation: with or without the prime, the left-hand side of the propagation equation is sensitive only to the changes in $h_{\alpha\beta}$ that hold τ_e constant.

The third of Eqs. (5.49) tells us that the field $h^{[0]}_{\alpha\beta}$ is orthogonal to its wave vector; and the fourth tells us it is trace-free.

Henceforth we shall ignore all post-geometric-optics corrections, and shall use $h^{[0]}_{\alpha\beta}$ as a high-accuracy approximation to $h_{\alpha\beta}$. Accordingly, we shall rewrite the equations (5.49) governing it as

$$k_\mu k^\mu = 0 \ , \qquad k_{\nu|\mu} k^\mu = 0 \ , \tag{5.50a}$$

$$h_{\alpha\beta} k^\beta = 0 \ , \qquad h_\alpha{}^\alpha = 0 \ , \tag{5.50b}$$

$$h_{\alpha\beta|\mu} k^\mu = -1/2 k^\mu{}_{|\mu} h_{\alpha\beta} \ . \tag{5.50c}$$

Notice that the curvature-coupling term $R^{\rm B}_{\mu\alpha\nu\beta} h^{\mu\nu}$, which appeared in the original propagation equation (5.33), is gone in the geometric-optics limit. It shows up only as a post-geometric-optics correction [the $\mathrm{O}(\epsilon^0)$ part of Eq. (5.48a), where the coupling of $h^{[0]}_{\alpha\beta}$ to the curvature helps generate the tiny correction $h^{[1]}_{\alpha\beta}$]. Because that curvature coupling is gone, in the geometric-optics limit $h_{\alpha\beta}$ satisfies the wave equation $\Box h_{\alpha\beta} = 0$; and because the waves' Riemann tensor is a linear sum of gradients of $h_{\alpha\beta}$ [Eq. (5.13)], it also satisfies the wave equation:

$$\Box R^{\rm GW}_{\alpha\beta\gamma\delta} = 0 \ . \tag{5.51}$$

This is the propagation equation (4.5) which was derived in a very different manner in Sec. 4.A and was used as the foundation for the description of gravitational radiation in Chap. 4. Note, further, that in the geometric-optics approximation the waves' Riemann tensor (5.13) takes the form

$$R^{\rm GW}_{\alpha\beta\gamma\delta} = 1/2(\ddot{h}_{\alpha\delta} k_\beta k_\gamma + \ddot{h}_{\beta\gamma} k_\alpha k_\delta - \ddot{h}_{\alpha\gamma} k_\beta k_\delta) \ . \tag{5.52}$$

This is identical in form to expression (4.27), except that here the field used is the geometric-optics approximation to the trace-free, Lorentz-gauge metric perturbation $h_{\alpha\beta}$, while in Chap. 4 the field used was the "gravitational-wave field" $h^{\rm GW}_{\alpha\beta}$. Recall that in Chap. 4 there was a separate gravitational-wave field $h^{\rm GW}_{\alpha\beta}$ associated with each nearly Lorentz reference frame, but that all those fields, when inserted into (5.52), produced the same frame-invariant Riemann tensor. In fact, in any specific small region of spacetime, one can adjust $h_{\alpha\beta}$ to be the same as the $h^{\rm GW}_{\alpha\beta}$ of any desired nearly Lorentz frame there by a gauge change with a generator that has the geometric-optics form

$$\xi_\alpha = \xi_\alpha(\tau_e/\epsilon,\, x^\mu) \ , \quad \text{where } \xi_\alpha k^\alpha = 0 \text{ and } \xi_{\alpha|\mu} k^\mu = 0 \ . \tag{5.53}$$

This gauge change, in fact, has precisely the same effect as the transverse-traceless projection process introduced in Eq. (4.50). Correspondingly, in the chosen nearly Lorentz frame the gravitational-wave field is

$$h^{\rm GW}_{jk} = (h_{jk})^{\rm TT} \ , \tag{5.54}$$

where the superscript TT means "perform the TT projection process of Eq. (4.50)".

We are ready, now, to make contact with the vacuum, geometric-optics propagation laws presented in Sec. 5.A: The field (5.2i) constructed there is the solution $h_{\alpha\beta}$ of the equations of propagation (5.50). To verify this is straightforward, except for one detail: it is necessary to know that the cross sectional area ΔA of a bundle of rays obeys the differential equation

$$\Delta A_{|\mu} k^\mu = k^\mu{}_{|\mu} \Delta A \ . \tag{5.55}$$

This is proved, for example, in Exercise 22.13 of MTW.

Finally, we note that for this geometric-optics solution (5.2i) to the propagation equations, Isaacson's gravitational-wave stress-energy tensor (5.38) reduces to

$$T^{\rm GW}_{\alpha\beta} = \frac{1}{16\pi} \left\langle (\dot h_+)^2 + (\dot h_\times)^2 \right\rangle k_\alpha k_\beta = \frac{1}{16\pi r^2} \left\langle (\dot Q_+)^2 + (\dot Q_\times)^2 \right\rangle k_\alpha k_\beta \ , \tag{5.56}$$

in accord with an assertion made in Chap. 4 [Eq. (4.32)].

5.E. Interaction with Matter

In developing the geometric optics formalism for gravitational-wave propagation, we ignored the coupling of the waves to the matter and nongravitational fields through which they propagate; i.e., we introduced the "vacuum approximation". In this section we shall study the coupling to matter, and in the next section, the coupling to electromagnetic fields. These studies will show that the vacuum approximation is highly accurate in the real astrophysical universe: the coupling to matter and electromagnetic fields can change only slightly the properties of the propagating waves. The sole exception is for waves emerging from the Planck era of the big bang. Near the Planck era, individual elementary particles and gravitons were so energetic that they could interact significantly (Sec. 7.2 of Zel'dovich and Novikov, 1983).

When gravitational waves pass through matter, they can be absorbed and scattered, and can develop dispersion (frequency-dependent propagation speeds). In this section we shall study absorption and dispersion, and shall describe and give references for scattering.

In our studies of absorption and dispersion, initially we shall confine attention to matter which, before the waves arrive, is static and has isotropic stresses and self-gravity that is negligible on lengthscales somewhat larger than the waves' reduced wavelength. More specifically, we shall assume that in the region of spacetime we are studying there exists a local Lorentz frame of $g^{\rm B}_{\alpha\beta}$ with size $\mathcal{D} \gg \lambdabar$ in which

$$T^{00}_{\rm B} = \rho_{\rm B} \ , \quad T^{0j}_{\rm B} = 0 \ , \quad T^{jk}_{\rm B} = p_{\rm B} \delta^{ij} \ , \tag{5.57}$$

with $\rho_{\rm B}$ (the density) and $p_{\rm B}$ (the pressure) independent of time $t = x^0$ but possibly dependent on x^j; and we shall insist that throughout this frame the gravitational interactions of the matter are negligible. By "negligible interactions" I mean that (*i*) $g^{\rm B}_{\alpha\beta}$ can be approximated by $\eta_{\alpha\beta}$, and (*ii*) when the matter is perturbed on scales $\lesssim \mathcal{D}$, the gravitational influence of one element of matter on any other can be ignored. This rules out, for example, using our analysis to study the coupling of gravitational waves to quadrupolar normal modes of the earth; but it permits a study of coupling to short-wavelength sound waves inside the earth, to the quadrupolar modes of resonant-bar gravitational-wave detectors, and to primordial plasma in the early universe. Note that our local Lorentz frame must always be small compared with the background radius of curvature, $\mathcal{D} \ll \mathcal{R}$, since one can never approximate $g^{\rm B}_{\alpha\beta}$ by $\eta_{\alpha\beta}$ on scales of order $\mathcal{R}$.

Specific examples that we shall study are a homogeneous perfect fluid (subsection *a* below), a homogeneous viscous fluid (subsection *b*), an inhomogeneous elastic medium, e.g., the earth (subsection *c*), a medium made of a large number of quadrupolar oscillators (subsection *d*), and a plasma (subsection *e*).

We presume that gravitational waves propagate into our local Lorentz frame from very far away, and thus are plane fronted on the scale of our frame. To simplify our calculations, we shall describe the waves in a gauge that is TT as the waves enter our frame. Since, as we shall see, interaction with the matter has only a tiny effect on the waves, on the right-hand side of the wave equation (5.33) we can treat the waves as having the undisturbed TT form

$$\bar{h}_{00} = \bar{h}_{0j} = 0, \quad \bar{h}_{jk} = h^{\rm GW}_{jk}(t - n_j x^j) \; . \tag{5.58a}$$

Here n_j is a unit vector in the propagation direction, and $h^{\rm GW}_{jk}$ is the transverse-traceless gravitational-wave field

$$h^{\rm GW}_{jk} n^k = 0 \; , \quad h^{\rm GW}_{jk} \delta^{jk} = 0 \; . \tag{5.58b}$$

This form of $\bar{h}_{\alpha\beta}$ has $\bar{h} = 0$, and when contracted into the background stress-energy tensor (5.57) it gives $\bar{h}_{\alpha\beta} T_{\rm B}^{\alpha\beta} = 0$, thereby bringing the wave equation (5.33) into the form

$$\Box \bar{h}_{\alpha\beta} + 2R^{\rm B}_{\mu\alpha\nu\beta} \bar{h}^{\mu\nu} = -16\pi \mathcal{T}_{\alpha\beta} \; . \tag{5.59}$$

[Our original definition (5.7) of $\mathcal{T}_{\alpha\beta}$ was carefully crafted so that it alone would remain on the right-hand side of (5.59). If we had used the more naive definition $\mathcal{T}_{\alpha\beta} = T_{\alpha\beta} - T^{\rm B}_{\alpha\beta}$, then the right-hand side of (5.59) would have contained an additional term $16\pi p_{\rm B} \bar{h}^{\alpha\beta}$, thereby complicating our calculations but, of course, not changing their final physical conclusions.]

Notice that in (5.59) only the TT part of $\mathcal{T}_{\alpha\beta}$ can contribute to the physically measurable waves. All other parts (e.g., $\mathcal{T}_{00}$ and $\mathcal{T}_{jk} n^k$) produce changes in $\bar{h}_{\alpha\beta}$ that are pure gauge, i.e., that contribute nothing to the waves' Riemann tensor and thus can be removed by a gauge transformation.

In the equation of motion (5.42) for the matter, as on the right-hand side of the wave equation, we can use the undisturbed wave field (5.58). When Eqs. (5.58) and (5.57) are inserted into Eq. (5.42), all terms on the right-hand side are found to vanish, leaving only

$$\mathcal{T}^{\alpha\mu}{}_{|\mu} = 0 \; . \tag{5.60}$$

At first sight one might think that this equation of motion implies the waves have no influence at all on the matter. On the contrary, as we shall see, there is a coupling of waves and matter embodied in $\mathcal{T}^{\alpha\beta}$ and hence in (5.60). Equation (5.60) governs the dynamical response of the matter to that coupling, and (5.59) governs the response of the waves.

Although Eqs. (5.59) and (5.60) are valid only in a local Lorentz frame of size $\mathcal{D} \ll \mathcal{R}$, they can be used to study wave propagation on scales $\gtrsim \mathcal{R}$: All one need do is string a series of local Lorentz frames together along the route of the waves, and as the waves enter each frame transform them to that frame's TT gauge.

To make these remarks more concrete and to get physical insight into wave-matter coupling, we shall now study several specific situations.

a. Homogeneous perfect fluid

The coupling of gravitational waves to a homogeneous, perfect fluid has been studied by a number of researchers over the years. The analysis which I like most is that of Gayer and Kennel (1979). The following calculation is patterned on it.

For a homogeneous perfect fluid, the stress-energy tensor in the full spacetime has the general form

$$T^{\alpha\beta} = (\rho + p)u^\alpha u^\beta + pg^{\alpha\beta} \tag{5.61}$$

(see, e.g., Sec. 22.3 of MTW). Here u^α is the fluid 4-velocity, and ρ and p are the density and pressure as measured in the fluid's local rest frame. Our background stress-energy tensor (5.57) has this form with $u^0_{\rm B} = 1$ and $u^j_{\rm B} = 0$. Suppose that the fluid is perturbed slightly, so that a particle originally at location x^j gets moved to coordinate location $x^j + \xi^j$. This displacement produces a fractional increase in fluid volume $\delta V/V = \xi^j{}_{|j}$; and correspondingly (by energy conservation), the density changes by $\delta\rho = -(\rho_{\rm B} + p_{\rm B})\xi^j{}_{|j}$, and the pressure changes by $\delta p = -K\xi^j{}_{|j}$, where K is the fluid's bulk modulus. The time-dependent displacement ξ^j also produces a first-order change $\delta u^0 = 0$, $\delta u^j = \dot{\xi}^j$ in the 4-velocity, where the dot denotes $\partial/\partial t$. These perturbations, together with $\delta g^{\alpha\beta} = -h^{\alpha\beta}$, all contribite to the stress-energy perturbation

$$T^{\alpha\beta} - T^{\alpha\beta}_{\rm B} = (\delta\rho + \delta p)u^\alpha_{\rm B} u^\beta_{\rm B} + 2(\rho_{\rm B} + p_{\rm B})u^{(\alpha}\delta u^{\beta)} + \delta p g^{\alpha\beta}_{\rm B} - ph^{\alpha\beta a} \; . \tag{5.62}$$

Our definition (5.7a) of $\mathcal{T}^{\alpha\beta}$ has been carefully crafted so that the wave-dependent term $-ph^{\alpha\beta a}$ will drop out of it. More specifically, by combining Eqs. (5.62) and (5.7a) and inserting the above values of $\delta\rho$, δp, and δu^α, we obtain

$$\mathcal{T}^{00} = -(\rho_{\rm B} + p_{\rm B} + K)\xi^i{}_{|i} \; , \quad \mathcal{T}^{0j} = (\rho_{\rm B} + p_{\rm B})\dot{\xi}^j \; , \quad \mathcal{T}^{jk} = -K\xi^i{}_{|i}\delta^{jk} \; . \tag{5.63}$$

Notice that nowhere at all in this $\mathcal{T}^{\alpha\beta}$ is there any gravitational-wave field $\bar{h}^{\alpha\beta}$, and recall that there is no explicit appearance of the wave field in the fluid's equation of motion (5.60). Correspondingly, as Gayer and Kennel (1979) conclude (see also p. 420 of Grishchuk and Polnarev, 1980 and references therein; XXXXXXX), *a gravitational wave cannot couple to a homogeneous, perfect fluid*. This is true in two senses: (i) When a wave hits the fluid, it leaves $\xi^i = 0$ and $\mathcal{T}^{\alpha\beta} = 0$; and correspondingly, the wave propagates through the fluid in accord with the standard, vacuum propagation equation

$\Box\bar{h}_{\alpha\beta} + 2R_{\mu\alpha\nu\beta}\bar{h}^{\mu\nu} = 0$. (*ii*) When sound waves, governed by $\mathcal{T}^{\alpha\mu}{}_{|\mu} = 0$, propagate through the fluid, they carry a nonzero $\mathcal{T}_{\alpha\beta}$ [Eq. (5.63)] whose spatial part (at first order in the fluid displacement) is a pure trace. Correspondingly, they generate via (5.59), a $\bar{h}^{\alpha\beta}$ wave whose spatial part is pure trace, and hence is pure gauge: it can be removed by a subsequent gauge transformation. In other words, pressure waves in a homogeneous perfect fluid cannot radiate gravitational radiation, at first order in the fluid displacement. [KIP: HOW ABOUT HIGHER ORDERS? I THINK IT IS REPUTED TO VANISH THERE ALSO.]

b. Homogeneous, viscous fluid

If our homogeneous fluid has shear viscosity as well as pressure and density, then its full stress-energy tensor (5.61) is augmented by $-2\eta\sigma^{\alpha\beta}$, where η is the coefficient of shear viscosity and $\sigma_{\alpha\beta}$ is the fluid's rate of shear (the symmetric, trace-removed part of the gradient of the 4-velocity, projected orthogonal to the 4-velocity); see, e.g., Exercise 22.6 of MTW. For the unperturbed fluid, with 4-velocity $u^0_{\rm B} = 1$ and $u^j_{\rm B} = 0$, the shear vanishes and the stress-energy tensor $T^{\alpha\beta}_{\rm B}$ has the form (5.57) considered above. However, when the fluid undergoes the displacement ξ^j and a gravitational wave of the form (5.58) is passing, the fluid experiences a rate of shear

$$\sigma_{00} = \sigma_{0j} = 0 \ , \qquad \sigma_{jk} = \xi_{(j|k)} - 1/3\xi^i{}_{|i}\delta_{jk} + 1/2\dot{h}^{\rm GW}_{jk} \ . \tag{5.64}$$

(The term $1/2\dot{h}^{\rm GW}_{jk}$ arises from a connection coefficient $\Gamma^0{}_{jk}$ in the computation of the gradient of the 4-velocity.) Correspondingly, the stress-energy perturbation $\mathcal{T}_{\alpha\beta}$ has the standard form for a slightly perturbed viscous fluid in flat spacetime, augmented by the coupling term

$$\delta\mathcal{T}_{jk} = -\eta\dot{h}^{\rm GW}_{jk} \ . \tag{5.65}$$

This term has a nonzero TT part. Thus, it produces a genuine coupling of the fluid to the gravitational waves.

The influence of this term on the waves can be studied by inserting it onto the right-hand side of the wave equation (5.59), taking the TT part of that wave equation so as to get rid of all extraneous, pure-gauge parts, and dropping the tiny curvature-coupling term (which we shall study in Sec. 5.G below). The result is

$$\Box h^{\rm GW}_{jk} = 16\pi\eta\dot{h}^{\rm GW}_{jk} \ . \tag{5.66}$$

A straightforward solution shows that the time-derivative term on the right-hand side produces a damping of the waves: The amplitude of the waves dies out as $\exp(-l/2l_{\rm atten})$, and the energy dies out as $\exp(-l/l_{\rm atten})$, where l is the distance travelled and $l_{\rm atten}$ is the energy attenuation length

$$l_{\rm atten} = \frac{1}{32\pi\eta} \ . \tag{5.67}$$

By restoring the factors of G and c, i.e., converting to cgs units via Eqs. (4.3), we bring this attenuation length into the form

$$l_{\rm atten} = \frac{c^6/G}{32\pi\eta} = (4.2\times 10^{18}\ \text{lt yr})\left(\frac{1\,\text{poise}}{\eta}\right) . \tag{5.67$'$}$$

When we recall that 1 poise is 1 dyne cm sec^{-2}, and that the viscosity of water is about 0.01 poise, we recognize that the gravitational waves' viscosity-induced attenuation length is exceedingly long.

Where does the gravitational waves' energy go? Into the fluid, of course. The wave-induced rate of shear, $\sigma_{jk} = 1/2\dot{h}^{\rm GW}_{jk}$, working against the viscosity, produces heat, thereby increasing the background energy density of the fluid at a rate $\partial\rho_{\rm B}/\partial t = 2\eta\sigma_{jk}\sigma^{jk}$ (MTW, Exercise 22.7). This rate of heating is precisely equal to the rate of loss of gravitational-wave energy,

$$\frac{\partial\rho_{\rm B}}{\partial t} = -T^{0j}_{{\rm GW}|j} = \frac{T^{0j}_{\rm GW} n_j}{l_{\rm atten}} , \tag{5.68}$$

as one can readily verify using expression (4.33) for the gravitational-wave energy flux and expression (5.67) for the waves' attenuation length. Notice, moreover, that this energy-balance relation (5.68) is just the general law of energy-momentum conservation (5.37), specialized to the present situation.

In order to estimate the magnitude of the attenuation length (5.67), we must consider the microscopic, particulate nature of the fluid. If the fluid is made of particles that move with mean speed v and that scatter off each other after traveling, on average, a mean free path $s \ll \bar\lambda =$(inhomogeneity scale of the perturbed fluid), then kinetic theory dictates that $\eta \sim \rho_{\rm B} v s$; see, e.g., XXXX. If $s \gtrsim \bar\lambda$ the diffusion approximation which underlies the theory of viscous fluids breaks down, but one can show that the above formula for viscous heating remains valid in order of magnitude if we set $\eta \sim \rho_{\rm B} v s(\bar\lambda/s)^2$. These expressions for the viscosity η, together with the fact that the fluid produces a background radius of curvature $\mathcal{R} \sim 1/(\text{Riemann tensor due to fluid})^{1/2} \sim 1/(\text{Einstein tensor due to fluid})^{1/2} \sim 1/(\text{energy density } \rho_{\rm B} \text{ due to fluid})^{1/2}$, implies that

$$\frac{l_{\rm atten}}{\mathcal{R}} \sim \frac{\mathcal{R}}{\bar\lambda}\frac{1}{v}\max\left(\frac{\bar\lambda}{s}, \frac{s}{\bar\lambda}\right) . \tag{5.69}$$

The magnitudes of the terms on the right-hand side are $\mathcal{R}/\bar\lambda \gg 1$ by virtue of the definition of a gravitational wave, $1/v \geq 1$ since the fluid's particles cannot travel faster than light, and $max(\bar\lambda/s,\ s/\bar\lambda) \gtrsim 1$. Consequently, the attenuation length is always much larger than the background radius of curvature of spacetime $\mathcal{R}$ produced by the fluid. Since the size of the fluid cannot exceed by much the radius of curvature $\mathcal{R}$ (when it reaches a size a bit larger than $\mathcal{R}$, it curls space up into closure around itself), this means that *no viscous fluid can produce significant attenuation of gravitational radiation.*

For more detailed treatments of the interaction between a viscous fluid and gravitational waves see, e.g., Esposito (1971a,b), Papadopoulos and Esposito (1985), Szekeres (1971), Sec. 4.2 of Grishchuk and Polnarev (1980) [KIP: CHECK THESE REFERENCES].

c. Inhomogeneous, elastic medium

For an inhomogeneous, elastic medium (e.g., a resonant-bar gravitational-wave detector), the background stress-energy tensor $T_{\rm B}^{\alpha\beta}$ has the standard form (5.57), and the stress-energy perturbation $\mathcal{T}^{\alpha\beta}$ is that of a perfect fluid (5.61), augmented by a shear restoring force

$$\delta\mathcal{T}^{\alpha\beta} = -2\mu\Sigma^{\alpha\beta} \ , \tag{5.70a}$$

where μ is the shear modulus and $\Sigma^{\alpha\beta}$ is the shear (the time integral of the rate of shear $\sigma^{\alpha\beta}$. By time integrating expression (5.64) we obtain

$$\Sigma_{00} = \Sigma_{0j} = 0 \ , \qquad \Sigma_{jk} = \Sigma^{\xi}_{jk} + 1/2 h^{\rm GW}_{jk} \ , \tag{5.70b}$$

where Σ^{ξ}_{jk} is the part of the shear produced by the displacement ξ^i

$$\Sigma^{\xi}_{jk} = \xi_{(j|k)} - 1/3 \xi^i{}_{|i}\delta_{jk} \ . \tag{5.70c}$$

By augmenting (5.70) onto the $\mathcal{T}_{\alpha\beta}$ of Eq. (5.63) and then inserting that $\mathcal{T}^{\alpha\beta}$ into the equation of motion (5.60) and specializing to the nearly Newtonian regime $p_{\rm B} \ll \rho_{\rm B}$, $K \ll \rho_{\rm B}$, we obtain

$$\rho_{\rm B}\ddot{\xi}_j - (K\xi^i{}_{|i})_{|j} - 2(\mu\Sigma^{\xi}_{jk})^{|k} = \mu^{|k} h^{\rm GW}_{jk} \ . \tag{5.71}$$

This is the standard equation of motion for a slightly perturbed, inhomogeneous, nonrelativistic elastic medium, augmented by the gravitational-wave driving term $\mu^{|k}h^{\rm GW}_{jk}$. This equation was first derived by Dyson (1969) and was subsequently generalized and studied in more elegant ways by Carter and Quintana (1977) and Carter (1983). Notice that *the waves drive the medium only through inhomogeneities of its shear modulus*.

For an elastic body that is small compared to $\bar{\lambda}$ (e.g., a resonant-bar gravitational-wave detector), one can study the waves' influence using the TT-gauge equation of motion (5.71), or one can study it in the body's proper reference frame, where the full metric $g_{\alpha\beta} = g^{\rm B}_{\alpha\beta} + h_{\alpha\beta}$ has the form (4.39). In the proper reference frame, the body's equation of motion will be the same as (5.71) but with the driving term $\mu^{|k}h^{\rm GW}_{jk}$ replaced by the standard gravitational-wave driving term $-1/2\rho_{\rm B}\ddot{h}^{\rm GW}_{jk}x^k$ [Eqs. (4.12) and (4.15)]. The two driving terms look completely different, and they give different displacement functions ξ^j. The reason is that the spatial coordinates of the TT gauge wiggle dynamically relative to those of the proper reference frame, thereby producing

$$\xi^{\rm TT}_j = \xi^{\rm PRF}_j - 1/2 h^{\rm GW}_{jk} x^k \ . \tag{5.72}$$

If one wants to be able to rely on ordinary physical intuition, one should use spatial coordinates that are as rigid as possible: proper reference-frame coordinates. However, if one wants only to solve the equations of motion of the medium without refering to elementary physical intuition, one is free to use either approach, proper-reference-frame or TT. If the medium is large compared to a reduced wavelength (e.g., for studies of the interaction of kilohertz waves with the earth's crust), one is stuck: only the TT analysis [Eq. (5.71)] is valid. The proper-reference-frame analysis fails. See the discussion in Sec. 4.F.

Turn attention from the influence of the waves on the medium to the influence of a homogeneous, elastic medium on the waves. The shear stress $\mathcal{T}_{jk} = -\mu h^{\rm GW}_{jk}$, acting back on the waves, produces dispersion; in addition, if there is viscosity, it will give rise to a shear stress $\mathcal{T}_{jk} = -\eta \dot{h}^{\rm GW}_{jk}$ [Eq. (5.65)] that damps the waves. We can see this quantitatively with the help of the waves' propagation equation (5.59). By (*i*) inserting into that propagation equation the above elastic and viscous contributions to $\mathcal{T}_{jk}$ along with the fluid contributions (5.63), (*ii*) taking the transverse-traceless part of the resulting equation [in accord with the paragraph following Eq. (5.59)], and (*iii*) ignoring the tiny curvature-coupling term, we obtain

$$\Box h^{\rm GW}_{jk} = 16\pi(\mu h^{\rm GW}_{jk} + \eta \dot{h}^{\rm GW}_{jk}) \; . \tag{5.73}$$

For waves with the sinusoidal form $h^{\rm GW}_{jk} \propto e^{i(kz-\omega t)}$, Eq. (5.73) gives the dispersion relation

$$\omega^2 = k^2 + 16\pi(\mu - i\omega\eta) \; . \tag{5.74}$$

Since, as we shall see, $16\pi|\mu - i\omega\eta|$ is always extremely small compared to $\omega^2 \cong k^2 = 1/\bar{\lambda}^2$, this dispersion relation corresponds to (*i*) an attenuation of the waves with an energy attenuation length the same as for a viscous fluid [Eq. (5.67)]

$$l_{\rm atten} = \frac{1}{32\pi\eta} \; , \tag{5.75}$$

and (*ii*) propagation with phase velocity $v_{\rm ph} = \omega/k$ and group velocity $v_{\rm gp} = \partial\omega/\partial k$ given by

$$v_{\rm ph} = 1 + 8\pi\mu\bar{\lambda}^2 \; , \quad v_{\rm gp} = 1 - 8\pi\mu\bar{\lambda}^2 \; . \tag{5.76}$$

Ordinary solid bodies have $\eta \sim 10^3\,{\rm g\ cm^{-1}\ sec^{-1}}$ and $\mu \sim 10^{10}$ dyne cm^{-2} which, by virtue of $c = 3 \times 10^{10}\,{\rm cm\ sec^{-2}} = 1$ and $G/c^2 = 0.7 \times 10^{-28}$ cm/g $= 1$, is the same as $\eta \sim 10^{-36}$ cm^{-1} and $\mu \sim 10^{-39}\,{\rm cm^{-2}}$. Correspondingly, *when propagating through an ordinary solid such as the earth, the gravitational waves' attenuation length is* $l_{\rm atten} \sim 10^{34}$ cm $\sim 10^6$ *Hubble distances; and their phase and group velocities differ from the speed of light by fractional amounts* $\sim 10^{-24}(\bar{\lambda}/100\,{\rm km})^2$. *Thus, kilohertz-frequency gravitational waves would have to propagate through earth-type matter a distance* $l_{\rm disp} \sim 10^{24}\bar{\lambda}^{-1} \sim 10^{31}\,{\rm cm} \sim 10^3$ *Hubble distances in order for dispersion to cause their phase to slip by just one radian.*

More generally, because the velocity of propagation of shear waves in any elastic medium is $\simeq \sqrt{\mu/\rho_{\rm B}}$ and cannot exceed light speed, it should always be true that $\mu \lesssim \rho_{\rm B} \sim 1/\mathcal{R}^2$. This means that gravitational waves in *any* elastic medium will have their phase shifted by one radian due to dispersion only after traveling a distance

$$l_{\rm disp} \sim \frac{1}{\mu\bar{\lambda}} \gtrsim \mathcal{R}\frac{\mathcal{R}}{\bar{\lambda}} \; . \tag{5.77}$$

Since the medium cannot be much larger than $\mathcal{R}$, and since gravitational radiation by definition must have $\bar{\lambda}/\mathcal{R} \ll 1$, *dispersion in an elastic medium can never produce a slippage of phase by even one radian*. Similarly, as was shown in Eq. (5.69), *attenuation can never be significant in an elastic medium.*

d. Medium made of quadrupolar oscillators

Turn, next, to the attenuation of gravitational radiation by that type of medium which, so far as I am aware, is the most effective of all realistic media for absorbing gravitational radiation: a medium made of a large number of "quadrupolar oscillators" with internal damping. By "quadrupolar oscillator", I mean a solid body with size $L \lesssim \bar{\lambda}$ and with a normal mode of oscillation that has a quadrupolar shape. Examples include resonant-bar gravitational-wave detectors, planets like the earth, stars like the sun, neutron stars, and black holes.

We can estimate the attenuation length in such a medium as $l_{\rm atten} = 1/n\sigma$, where σ (not to be confused with the rate of shear of a fluid) is the cross section for an individual oscillator to absorb gravitational-wave energy and n is the number density of oscillators. To evaluate the cross section σ we compute the response of an oscillator to a passing, monochromatic gravitational wave. That response, in the oscillator's proper reference frame (*not* in TT gauge), is governed by the equation

$$\frac{d^2\delta L}{dt^2} + \frac{1}{\tau_*}\frac{d\delta L}{dt} + {\omega_o}^2\delta L \simeq L\frac{d^2h_+}{dt^2}; \tag{5.78}$$

cf. Exercise 37.10 of MTW. Here δL is the generalized coordinate of the oscillator's quadrupolar normal mode, so normalized as to be equal to the physical displacement of a representative piece of the oscillator's surface, and h_+ is the gravitational-wave field evaluated at the oscillator's location. (Since the oscillator is smaller than a reduced wavelength of the waves, $L \lesssim \bar{\lambda}$, spatial variations of h_+ are unimportant inside the oscillator.) The left-hand side of (5.78) is the standard harmonic-oscillator equation for the normal mode's generalized coordinate, and the right-hand side is an order-of magnitude estimate of the gravitational-wave driving force, based on Eqs. (4.12), (4.16) and (4.25). The parameter ω_o is the eigenfrequency of the normal mode, τ_* is its damping time, L is the oscillator's linear size, and below we shall denote by M the oscillator's mass and by $Q \equiv \omega_o\tau_*/\pi$ the normal mode's "quality factor".

The approximate equation of motion (5.78) and the order-of-magnitude analysis that follows are valid for bodies with strong self-gravity (e.g., neutron stars and black holes) as well as weak (the earth or a resonant-bar gravitational-wave detector).

By setting $h_+ = he^{-i\omega t}$ (where $\omega = 1/\bar{\lambda}$ is the wave's angular frequency and h is its amplitude), solving (5.78) for δL, then computing the energy of oscillation $E_{\rm osc} \sim 1/2M{\omega_o}^2\times$ (amplitude of $\delta L)^2$, and then multiplying by $1/\tau_*$, we obtain the rate that energy is fed into internal damping of the oscillator—a rate which, in this steady-state situation, must equal the power absorbed by the oscillator from the gravitational wave, $P_{\rm abs}$. By equating this $P_{\rm abs}$ to the product of the cross section σ and the waves' energy flux $(1/16\pi)\omega^2h^2$ [Eq. (4.34)], we obtain the following expression for the oscillator's cross section

$$\sigma(\omega) \sim \frac{ML^2\omega^2/\tau_*}{\omega^2 - {\omega_o}^2 + (2\tau_*)^2}\ . \tag{5.79}$$

We shall return to the frequency dependence of this expression in Chap. 10, when discussing gravitational-wave detectors. For now, in order to put an upper limit on the effects of absorption, we shall suppose that the waves are precisely on resonance, $\omega = \omega_o$. Then σ achieves its maximum value of

$$\sigma_{\max} \sim \frac{M}{L}\frac{L}{\bar{\lambda}}QL^2 \ , \tag{5.80}$$

and the attenuation length achieves its minimum possible value, $l_{\text{atten}} = 1/n\sigma_{\max}$. Expressed in terms of the radius of curvature of the background spacetime produced by the attenuating oscillators, $\mathcal{R} \sim 1/\rho^{1/2} = (1/nM)^{1/2}$, this works out to be

$$\frac{l_{\text{atten}}}{\mathcal{R}} \sim \frac{\mathcal{R}}{\bar{\lambda}}\left(\frac{\bar{\lambda}}{L}\right)^2 \frac{1}{Q} \ . \tag{5.81}$$

The first term, $\mathcal{R}/\bar{\lambda}$ is $\gg 1$ by the definition of a gravitational wave. The second term, $(\bar{\lambda}/L)^2$, is equal to the speed of light divided by the velocity of sound inside the oscillating body, squared, and thus is always $\gtrsim 1$. [For a black hole $(\bar{\lambda}/L)^2 \sim 1$, for a neutron star $(\bar{\lambda}/L)^2 \sim 10$, for a white dwarf $(\bar{\lambda}/L)^2 \sim 10^4$, for normal stars like the sun $(\bar{\lambda}/L)^2 \sim 10^6$, for the earth or a resonant-bar gravitational-wave detector $(\bar{\lambda}/L)^2 \sim 10^{10}$.] By contrast, the third term, $1/Q$, is generally $\ll 1$. However, for realistic astrophysical situations $1/Q$ is never sufficiently small to make up for the large product of the first two terms. Moreoever, when the resonant angular frequencies of the many oscillators are spread out randomly over a band $\Delta\omega_o \sim \omega_o$, only a fraction $\sim 1/Q$ of the oscillators will be near enough resonance to have $\sigma \sim \sigma_{\max}$; and correspondingly the factor $1/Q$ in (5.81) will be replaced by unity. Thus, *although, in principle, gravitational waves could be attenuated significantly by a medium of quadrupolar oscillators, in realistic situations the attenuation length greatly exceeds the radius of curvature $\mathcal{R}$ produced by the oscillators; and thus, as in the case of a viscous medium, attenuation cannot be astrophysically important*.

A simple extension of this argument shows that dispersion also cannot be significant: the total phase shift, due to dispersion, in traveling the distance $\mathcal{R}$ through a realistic astrophysical medium is generally $\ll 1$; see Sec. 2.4.3 of Thorne (1983). It is instructive and fun, however, to imagine and study theoretically an unrealistic form of matter ("respondium") with such strong dispersion that it actually reflects gravitational waves (Press 1979).

The specific problem of the absorption and scattering of gravitational waves by black holes has been studied in extensive detail. Those studies reveal a great richness of scattering phenomena (superradiant scattering, "glories", rotation of the waves' polarization, ...); see De Logi and Kovacs (1977), Matzner and Ryan (1978), Matzner *et al*. (1985), Futterman, Handler, and Matzner (1988), and references therein. For studies of absorption and scattering by neutron stars and other stars see Linet (1984). For a study of how the energy levels of atoms are shifted by the tidal gravity of passing gravitational waves [fractional shifts no larger than $h \times$ (size of atom)$^2/\bar{\lambda}^2$, which is almost certainly too small in the real universe to be measurable], see Leen, Parker, and Pimentel (1983).

e. Plasmas

The most astrophysically ubiquitous of all media is a magnetized plasma. Careful studies of the propagation of gravitational waves through plasmas are only now in process

(Ehlers, Prasanna, and Breuer, 1986; Bondi and Pirani, 1988), and there are no definitive results at this writing, with one exeption: Gayer and Kennel (1979), and Grishchuk and Polnarev (1980, p. 420 – KIP: CHECK) have shown that in an unmagnetized plasma, as in the media studied above, dispersion is so extremely small that there is no hope for particles with mass to "ride along with the crests" of a gravitational wave and produce Landau damping. *Landau damping of gravitational radiation in a plasma is totally negligible*. This result, and intuition built up from the above calculations for other media, make me rather confident that a magnetized plasma cannot have any significant influence on gravitational waves that propagate through it. Nevertheless, there are likely to be a number of fascinating but tiny effects of the waves on the plasma and the plasma on the waves.

5.F. Interaction with Electromagnetic Fields

Turn attention from propagation of gravitational waves through matter, to propagation through electromagnetic fields.

Because gravitational and electromagnetic waves should propagate with the same speed, they can interact in a coherent way (Gertsenshtein, 1962). The interaction is so weak, however, that a substantial transformation of one into the other requires propagation over a distance of order the radius of curvature $\mathcal{R}$ of the background spacetime which their own energy density produces. Thus, such coherent interaction is not likely ever to be important in the real universe—except possibly in gravity-wave detectors; see Sec. 12.A below. For a review of the extensive literature on this subject see Grishchuk and Polnarev (1980). For a brief pedagogical discussion see Sec. 17.9 of Zel'dovich and Novikov (1983).

The situation in which one might have expected the strongest resonant interaction is for electromagnetic and gravitational waves propagating parallel to each other through an otherwise empty spacetime. Remarkably, in this situation there is no interaction whatsoever (Braginsky and Grishchuk, 1977 KIP: LEONYA CLAIMS THIS PAPER DOES NOT EXIST AND SAYS TO CITE BRAGINSKY ET AL 1973; Grishchuk, 1977b – KIP: LEONYA SAYS THIS REFERENCE IS NOT NECESSARY). It is easy to see why, for the special issue of whether the stress-energy tensor of a propagating electromagnetic wave generates a gravitational wave propagating along with it. That stress-energy tensor, in a local Lorentz frame of the background metric, has the form

$$T^{00} = T^{0z} = T^{z0} = T^{zz} = \frac{{E_o}^2}{4\pi} \cos^2 \omega(t-z) \; , \tag{5.82}$$

where E_o is the amplitude of oscillation of the electric field and the wave is presumed to be plane polarized and to propagate in the z-direction. The oscillatory part of $T^{\mu\nu}$, which one might expect to generate gravitational waves, is

$$\mathcal{T}^{00} = \mathcal{T}^{0z} = \mathcal{T}^{z0} = \mathcal{T}^{zz} = \frac{{E_o}^2}{8\pi} \cos 2\omega(t-z) \; . \tag{5.83}$$

When fed into the wave equation (5.33) (with the coupling of $\bar{h}_{\alpha\beta}$ to the background fields $R^{\rm B}_{\alpha\beta\gamma\delta}$ and $T^{\rm B}_{\alpha\beta} \equiv \langle T_{\alpha\beta} \rangle$ ignored because it can be important only over the extremely long lengthscale $\mathcal{R}$), this $\mathcal{T}^{\alpha\beta}$ produces a trace-reversed metric perturbation $\bar{h}_{\alpha\beta}$ that propagates along with the electromagnetic waves at the speed of light, growing linearly with the distance traveled. However, this $\bar{h}_{\alpha\beta}$, like the $\mathcal{T}_{\alpha\beta}$ that generates it, is purely longitudinal; i.e., it has 00, $0z$, $z0$, and zz components but no components in transverse directions. This implies that this wavelike $\bar{h}_{\alpha\beta}$ is "pure gauge": it can be transformed to zero by a change of gauge, as one can see most easily by noting that its transverse-traceless projection (4.50) vanishes and therefore that it is associated with a vanishing gravitational-wave field $h^{\rm GW}_{jk}$.

To achieve a resonant interaction between parallely propagating electromagnetic and gravitational waves, one can send them through a time-independent ("DC") electric or magnetic field (Gertsenshtein, 1962). As a simple example, let a pure electromagnetic wave, with electric field $\mathbf{E} = E_o \cos\omega(t-z)\mathbf{e}_x$, propagate from vacuum at $z<0$ into a region of homogeneous, DC field $\mathbf{E} = E_{\rm DC}\mathbf{e}_x$ at $z>0$; and assume that $E_{\rm DC} \gg E_o$. Then the beating of the wave's electric field against the DC field produces an oscillating stress-energy tensor which has (among others) the transverse components

$$\mathcal{T}_{xx} = -\mathcal{T}_{yy} = \frac{E_{\rm DC}E_o}{4\pi}\cos\omega(t-z) \quad \text{for } z>0 \,. \tag{5.84}$$

Fed into the wave equation (5.33) (with background-coupling effects neglected as above), these $\mathcal{T}_{\alpha\beta}$ generate a trace-reversed metric perturbation

$$\bar{h}_{xx} = -\bar{h}_{yy} = \frac{2E_{\rm DC}E_o}{\omega}\, z\, \sin\omega(t-z) \quad \text{for } z>0 \,. \tag{5.85}$$

The TT projection of this $\bar{h}_{\alpha\beta}$ [Eq. (4.50)] gives a gravitational-wave field

$$h_+ = \frac{2E_{\rm DC}E_o}{\omega}\, z\, \sin\omega(t-z) \,, \quad h_\times = 0 \quad \text{for } z>0 \,. \tag{5.86}$$

The amplitude of this wave grows linearly with the distance z travelled, and correspondingly its energy density grows quadratically. It is easy to verify that the ratio of the energy in the growing gravitational wave to that in the original electromagnetic wave is of order $(z/\mathcal{R})^2$, where $\mathcal{R}$ is the background radius of curvature produced by the DC field. By virtue of total energy conservation, there is a back-action on the electromagnetic wave which saps energy from it as the gravitational wave grows. A detailed analysis (XXXXX) shows that, once all the wave energy has been fed into the gravitational wave, the beating of that gravitational wave against the DC field begins to regenerate the electromagnetic wave. The wave energy thereafter sloshes back and forth between electromagnetic and gravitational waves, with a sloshing lengthscale of order $\mathcal{R}$. [KIP: SEE GRISHCHUK'S REFS. PRIVATELY HE SAYS ZEL'DOVICH JETP 65, 1311 (1973), SOV PHYS JETP 38, 652 (1974); ALSO GERLACH PRL 32, 1023 (1974)]

Because the background field cannot have a longitudinal extent much larger than $\mathcal{R}$ (see above), the sloshing cannot continue for more than a few cycles. And, for realistic laboratory or astrophysical parameters, there cannot be any sloshing at all; there is only a fractionally tiny interconversion of one wave type into the other.

A specific, well-studied example of the above process is the interconversion of gravitational and electromagnetic waves as they propagate near an electrically charged black hole (XXXXXX). Another cute application is a proof that in principle (though never in practice) this interconversion, plus interaction of the electromagnetic waves with thermal matter, can be used to thermalize initially nonthermal gravitational radiation (Garfinkle and Wald, 1985). A third application is to gravitational-wave detectors in which a strong, DC field (e.g., the electric field in a parallel-plate capacitor) is driven by incoming gravitational waves to produce a tiny amount of electromagnetic radiation (Braginsky *et al.*, 1973; Sec. 12.A, below).

Interactions between gravitational and electromagnetic waves can be catalyzed not only by a background, DC electromagnetic field, but also in other ways: (*i*) Parallely propagating electromagnetic and gravitational waves can be coupled by a dielectric medium, but the coupling is proportional to $n - 1$ where n is the dielectric's index of refraction, and the coupling is so weak that it probably is of no practical interest (Grigor'ev, 1982). (*ii*) When electromagnetic and gravitational waves propagate through vacuum in nonparallel directions, they interact weakly. For example, if the wavelength of the gravitational wave is much longer than that of the electromagnetic wave, then the gravitational wave can produce electromagnetic polarization (Polnarev, 1985), it can rotate the plane of electromagnetic polarization (Cruise, 1983), and it can produce fluctuations in the frequency, intensity, and direction of the electromagnetic wave (Zipoy, 1966; Bergman, 1971; Bertotti and Catenacci, 1975; Adams, Hellings, and Zimmerman, 1984). Such interactions have only minuscule influence on either the gravitational or the electromagnetic wave; but, thanks to high-precision technology, the tiny electromagnetic effects can be used in a variety of promising ways for gravitational-wave detection (Sec. 12.C).

5.G. Catalog of Vacuum Wave-Propagation Effects

Having seen that the interaction of gravitational waves with matter and electromagnetic fields is almost never significant, we now return to the vacuum approximation. There are a number of peculiarities of vacuum wave propagation. Some show up in the equations of geometric optics, while others are removed by the approximations that underlie the geometric-optics formalism. In this section we shall discuss the most interesting and important of the vacuum propagation phenomena, and we shall examine their relationship to geometric optics and their relevance to the real universe.

a. Scattering by background curvature, and tails of waves

Gravitational waves can sometimes encounter regions of spacetime where the background radius of curvature becomes comparable to or shorter than the reduced wavelength, $\mathcal{R} \lesssim \bar{\lambda}$. When this happens, not only do the shortwave and geometric optics formalisms break down, but the very concept of a gravitational wave becomes meaningless. Nevertheless, one can continue to analyze the evolution of the metric perturbations $h_{\alpha\beta}$ using the formalism of Sec. 5.B.

Of particular interest when $\mathcal{R} \lesssim \bar{\lambda}$ is the scattering of the perturbations by the background curvature. Such scattering shows up in the vacuum wave equation $\Box \bar{h}^{\alpha\beta} +$

$2R^{\rm B}_{\mu\alpha\nu\beta}\bar h^{\mu\nu} = 0$ [Eq. (5.24)] as a result not only of the curvature-coupling term, but also of the influence of the background on the wave operator $\Box$. This scattering is very important in some sources of waves, as the waves are trying to form. For example, it is responsible for the normal-mode vibrations of black holes (Press, 1971; Goebel, 1972; Sec. 6.J), and it leads to the formation of "tails" of the waves in a source's near zone (Price, 1972a,b; Thorne, 1972; Cunningham, Price, and Moncrief, 1979) and to radiative tails in the wave zone (Leaver, 1986; Blanchet, 1987b; XXXX)—which, though they are very important in principle (Newman and Penrose, 1965; Blanchet and Damour, 1988a), are not likely ever to be observationally important. Remarkably, in a homogeneous cosmological model filled with perfect fluid with vanishing pressure or with pressure equal to 1/3 the energy density, there is no backscatter off the spacetime curvature whatsoever (Janis, 1985; Gayer and Kennel, 1979).

b. Parametric amplification by background curvature

In regions of a dynamical spacetime (e.g., the expanding universe) in which the characteristic wavelength$\bar\lambda$ of gravitational waves is larger than or comparable to the background radius of curvature $\mathcal{R}\bar\lambda \gtrsim \mathcal{R}$, the waves can be parametrically amplified by interaction with the dynamical background (Grishchuk, 1974, 1975a,b, 1977a; Grishchuk and Polnarev, 1980; Allen, 1988; Sec. 9.D). Viewed quantum mechanically, the interaction causes stimulated emission of new gravitons (XXXX). Viewed classically, the interaction can be analyzed using the standard, perturbed, vacuum Einstein equation $\Box\bar h^{\alpha\beta} + 2R^{\rm B}_{\mu\alpha\nu\beta}$ $= 0$; and the parametric amplification comes about because of the time dependence not only in $R_{\mu\alpha\nu\beta}$ but also in the background connection coefficients that appear in the wave operator. Such parametric amplification may well have enabled the expansion of the universe to take gravitational vacuum fluctuations that emerged from the Planck era of the big-bang, and enlarge them into a strong, stochastic background of gravitational waves today; see Sec. 9.D below.

c. Gravitational focusing

Lumps of background curvature associated with black holes, stars, star clusters, and galaxies will focus gravitational waves in precisely the same manner as they focus electromagnetic waves; i.e., they act as "gravitational lenses" for the waves. Just as this focusing is observationally important for the light and radio waves from a few very distant quasars, so it might also be important for distant discrete sources of gravitational waves. Focusing by the sun, in the case of waves with sufficiently short wavelength, can be significant, but not at earth; the focal point lies farther out in the solar system, near the orbit of Jupiter (Cyranski and Lubkin, 1974). Gravitational focusing shows up clearly in the waves' geometric-optics propagation: A bundle of rays, along which the waves are propagating, is focussed gravitationally. This causes the bundle's cross sectional area ΔA and radial variable r [Eq. (5.2d)] to decrease, and the h_+ and $h_\times$ of Eq. (5.2e) to increase.

d. Diffraction

Near the focal point of a gravitational lens, the radii of curvature of the wave fronts are no longer huge compared to the waves' reduced wavelength. As a result, the waves cease to propagate along null rays and begin to diffract, thereby lessening the strength of the focusing. The analysis of this is no different for gravitational waves than for electromagnetic or scalar waves, since polarization plays no important role. The analysis can be carried out using the flat-spacetime wave equation $\Box h_{\alpha\beta} = 0$ in a nearly Lorentz frame of the focal region. One switches from geometric optics to this wave equation as the waves near the focal point. Then, once the waves are well past the focal point, one can return to geometric optics. One thereby finds that diffraction almost completely wipes out the effects of gravitational focusing, unless the waves' reduced wavelength $\bar{\lambda}$ is small compared to the lens's gravitational radius $2GM/c^2 = 3\,\mathrm{km}(M/M_\odot)$. (Here M is the lens's mass.) This criterion applies whether the lens is a black hole, or a star like the sun, or a galaxy. For an order of magnitude discussion see, e.g., Sec. 2.6.1 of Thorne (1983); for full details see Bontz and Haugan (1981).

e. Nonlinear wave-wave coupling (frequency doubling, etc.)

Because general relativistic gravity is nonlinear, there is a nonlinear self-coupling of gravitational waves ("wave-wave coupling"). In principle this leads to such nonlinear conversion processes as frequency doubling. One can compute the effects of nonlinearities using an explicit version of Eq. (5.32c). Such a calculation shows that in practice wave-wave coupling effects are not important in regions where the waves *are* waves (where $\bar{\lambda} \ll \mathcal{L}$). This is because, in such regions, the dimensionless amplitude h of the waves is small compared to unity [Eq. (5.41) above]. For details see Sec. 2 of Thorne (1985). However, in idealized situations where $\bar{\lambda}$ becomes temporarily $\sim \mathcal{L}$ and h becomes temporarily of order unity, significant frequency doubling can occur. An example is the focusing of beams of gravitational radiation into a region so small (of order $\bar{\lambda}$) and with such great beam intensity ($h \sim 1$) that the focused radiation almost but not quite forms a black hole. Upon diffracting and reexploding, the radiation shows signficant frequency doubling (Abrahams, 1987).

Even when the waves' amplitude h becomes of order unity (and, as a result, the distinction between background curvature and wave curvature begins to break down), there is no evidence in any calculations to date for a steepening of the waves to form gravitational shocks (discontinuities in the waves' Riemann tensor). This remains true even when one adds additional nonlinearities to the Einstein field equation by augmenting the Einstein-Hilbert Lagrangian by terms quadratic in the curvature tensor (Tomimatsu, 1987).

f. Generation of background curvature by the waves

The generation of background curvature by the stress-energy of the waves [Isaacson, 1968b; MTW Sec. 35.15; Eq. (5.38) above] is important in cosmological models in any epoch when the waves are sufficiently strong that their energy density is comparable to that of matter, or larger; see, e.g., Hu (1978) and Chap. 17 of Zel'dovich and Novikov (1983). It is also important in a gravitational "geon"—i.e., a bundle of gravitational waves that is held together by its own gravitational pull on itself (Wheeler, 1962; pp. 409–438 of Wheeler, 1964; Brill and Hartle, 1964). But geons surely do not exist in the real universe. There is no reason to expect them to form, and if they did form they would quickly disrupt due to a large-scale, global instability (XXXXXX). Nevertheless, geons are important theoretical entities: they are useful for exploring issues in fundamental physics.

g. Nonlinear effects in collisions of gravitational waves

The head-on collisions of precisely planar gravitational waves (with infinite transverse extent) have been studied using exact, rather than approximate solutions of the vacuum Einstein field equation; see Sec. 5.H below. These solutions, and exact theorems about them, reveal a number of interesting nonlinear features: (*i*) The background curvature produced by each wave acts as a lens to focus the other wave. Because the waves have infinite transverse extent, diffraction does not occur, and each focussed wave converges onto its focal plane to produce a spacetime singularity (Kahn and Penrose, 1971; Szekeres, 1972; Nutku and Halil, 1977; Tipler, 1980; Matzner and Tipler, 1984). (*ii*) The singularity has an "inhomogeneous Kasner structure" with infinite tidal squeezing along two spatial axes and infinite stretching along the third (Yurtsever, 1987a, 1988). (*iii*) For special forms of the pre-collision waves, some or all of the singularity gets replaced by a "Cauchy horizon". However, those special forms are nongeneric: arbitrarily weak changes in them cause the collision to produce an all-embracing singularity rather than a Cauchy horizon (Chandrasekhar and Xanthopoulos, 1986, 1987; Yurtsever, 1987a, 1988).

Exact theorems show that, in the more realistic case of colliding waves that are almost planar but die out slowly at large transverse distances, if the transverse size is sufficiently large compared to the initial wave amplitude, then the focusing still drives the amplitude up far enough, before diffraction can act, to make nonlinear effects take over and produce singularities (Yurtsever, 1987b, 1988). If the "cosmic censorship conjecture" is correct, those singularities must be hidden inside one or two black-hole horizons; but it has not yet been possible to determine whether this is so. Unfortunately, the wave size required to produce a singularity is so huge that wave-wave collisions almost certainly do not form singularites in the real universe, except possibly near the big bang (Yurtsever, 1987b, 1988).

Collisions of gravitational waves produce not only focusing, but also a rotation of the polarization axes of one wave by the gravitational action of the other—a phenomenon discovered in collisions of cylindrical waves by Piran and Safier (1985).

5.H. Asymptotic Analyses and Exact Solutions

We conclude this chapter with a brief description of studies of gravitational-wave propagation in special circumstances.

a. Asymptotic analyses

Much has been learned about the geometric properties of gravitational radiation by studying the idealized problem of the propagation of waves outward from a source that resides alone in an otherwise empty and asymptotically flat spacetime: In analyses that were central to building up confidence in our understanding of gravitational waves, Bondi (1960), Bondi, van der Burg, and Metzner (1962), and Sachs (1962, 1963) expanded the spacetime curvature along the outgoing light cone in inverse powers of the distance to the source. Their expansions, carefully formulated and combined with conformal transformations that bring “infinity” in to finite locations (Penrose 1963a,b) revealed an elegant asymptotic structure for waves in asymptotically flat spacetime. The study of this asymptotic structure has been pursued with vigor in recent years; for reviews and references see Newman and Todd (1980), Ashtekar (1981, 1984), Walker (1983), Schmidt (1979, 1986), Hobill (1984), Penrose and Rindler (1986), Friedrich (1986), Blanchet and Damour (1986), Blanchet (1987a), Winicour (1988), and Ashtekar and Schmidt (1990).

b. Exact solutions to the vacuum Einstein field equation

Much insight into gravitational radiation has come from exact solutions to the vacuum Einstein field equation.

One broad class of exact solutions, called “boost-rotation symmetric spacetimes”, describes an idealized class of gravitational-wave sources that radiate into a (nearly) asymptotically flat spacetime.

The sources are axially symmetric and invariant under a Lorentz-like “boost”. They include such idealized configurations as two black holes with a spring between them, which forces them to accelerate uniformly away from each other (the “C-metric”, discovered by Levi-Civita, 1918 and explored and interpreted physically by Kinnersley and Walker, 1970; Bonnor, 1983; and others). The spacetime into which these special sources radiate is asymptotically flat (like Minkowskii spacetime), except for a weak, cosmic-string-type structure (circumference, divided by $2\pi\times$ radius, equal to a constant slightly less than one) along the direction of acceleration (the symmetry axis). For the general theory and reviews of boost-rotation-symmetric spacetimes see Bi **c** ak (1968); Bi **c** ak, Hoenselaers and Schmidt (1983); Bi **c** ak and Schmidt (1984); and Bi **c** ak (1985, 1988).

Although there are no other known exact solutions describing waves that propagate out into asymptotically flat spacetime, there is a general formalism for a much broader and more realistic class of solutions—a formalism sufficiently powerful to permit proof of interesting theorems and to give promise of ultimately producing exact solutions. This formalism, due to Robinson and Trautman (1962), describes wave-carrying spacetimes whose rays are “geodesic and hypersurface orthogonal” (properties shared by the rays of geometric optics) and in addition are free of shear. [KIP: CHECK - HOW CAN THEY BE SHEAR-FREE?] Among the important, rigorous theorems that have been proved for such spacetimes is one which says that the waves must die out at late times, leaving behind the Schwarzschild spacetime of a nonrotating black hole (Forster and Newman, 1967; Lucacs *et al.*, 1984). For reviews and references on the Robinson-Trautman formalism see, e.g., Kramer *et al.* (1980) and Schmidt (1987). For a first step in generalizing the Robinson-Trautman theory to spacetimes whose rays have “twist”, see Chinea (1988).

Nonlinear interactions of gravitational waves with themselves and each other have been studied extensively using exact solutions which are plane symmetric or cylindrically symmetric—and, thus, which extend outward infinitely far in one or two transverse directions. Powerful, soliton-theoretic techniques for generating such exact solutions have been devised by Belinsky and Zakharov (XXXX); and other solution-generating techniques have been developed by Chandrasekhar (1986 and refs. therein) and by Ernst, Garcia, and Hauser (1988). For applications of these techniques see, e.g., Cespedes and Verdaguer (1987); Garriga and Verdaguer (1987); and Ferrari, Ibanez, and Bruni (1987). The physical properties of cylindrical gravitational waves have been explored, e.g., by Einstein and Rosen (1936) and Weber and Wheeler (1957), who pioneered the subject; and by Thorne (1965a,b), Schmidt (1981), Piran, Safier, and Katz (1986), Chandrasekhar and Ferrari (1987), and others. For detailed studies of planar gravitational waves see Rosen (1937), Bondi, Pirani, and Robinson (1959), and Ehlers and Kundt, 1962 (the pioneering papers), and, more recently, the papers cited in Sec. 5.G.g in connection with gravitational-wave collisions.

Insight into cosmological gravitational waves originating in the early universe comes from a family of exact solutions generated by the Belinsky-Zakharov (XXXX) technique. These solutions describe universes which, at early times, contain "frozen-in" inhomogeneities. As the cosmological horizon expands and becomes larger than the inhomogeneities' reduced wavelength $\bar{\lambda}$, the inhomogeneities unfreeze and are smoothly transformed into gravitational radiation propagating dynamically through an otherwise homogeneous universe. For specific solutions of this type see Carr and Verdaguer (1983), Belinsky and Francaviglia (1984), and Adams, Hellings, and Zimmerman (1985).

A final type of exact solution which is useful for insight is the extreme limit of geometric optics, where the wavelength becomes so short that the radiation is compacted into a *gravitational shock wave*. For the exact theory of gravitational shocks see, e.g., Pirani (1957), Papapetrou (1977), and references therein.