

**WEEK 9: SUPERMASSIVE BLACK HOLES – PHENOMENOLOGY & GW SOURCES;
GW'S FROM BIG BANG – AMPLIFICATION BY INFLATION****Lectures 15 by Phinney, and 16 by Thorne****Reading Related to Sterl's Lecture:**

Not only are there no textbook treatments of this material, there are no review articles that cover all or even most of the issues in Sterl's lecture. The following articles cover the indicated topics. You are encouraged to browse or study a few of them. Of special interest is the paper by Begelman, Blandford and Rees (reference 2.a), which you are asked to explore in Exercise 3 below.

1. *Supermassive black holes in nuclei of galaxies*

- a. J. Kormendy & K. Gebhardt, "Supermassive black holes in nuclei of galaxies", astro-ph/0105230
- b. D. Merritt & L. Ferrarese, "Relationship of black holes to bulges", astro-ph/0107134
- c. P. Salucci et al, "The mass of black holes in the nuclei of spirals," 1999 MNRAS 307, 637

2. *Formation and evolution of supermassive black-hole binaries in merging galaxies*

- a. Begelman, Blandford & Rees 1980 Nature, 287, p. 307
- b. Milosavljevic & Merritt astro-ph/0103350
- c. Yu astro-ph/0109530

3. *Galaxy merger rates*

- a. *Observations*: Patton et al astro-ph/0109428 [section 8.5 p 24].
- b. *CDM theory*: Murali et al astro-ph/0106282
- c. *More on CDM theory*: Khochfar & Burkert astro-ph/0105383

4. *Galaxy mergers:*

- a. Arp: Atlas of Peculiar Galaxies [online at http://nedwww.ipac.caltech.edu/level5/Arp/Arp_contents.html]
- b. Toomre & Toomre 1972 ApJ, 178, 623
- c. Barnes & Hernquist 1992 ARA&A 30, 705

5. *Rates for inspiral of compact bodies into supermassive black holes*

- a. Hills & Bender 1995 ApJ, 445, L7
- b. Sigurdsson & Rees 1997 MNRAS 284, 318

- c. Miralda-Escude & Gould astro-ph/0003269
 - d. M. Freitag, Geneva Observatory PhD Thesis [enquire within room 152 Bridge]
 - e. Phinney, 06-08-01 entry on <http://www.its.caltech.edu/cajagwr/scripts/seminars.html>
6. *Detectability of waves from inspiral of compact bodies into supermassive black holes*: Finn & Thorne Phys. Rev. D 62, 124021 (2000)

Reading Related to Kip's Lecture:

7. *Overview of gravitational waves from supermassive black hole binaries, from inspiral of compact bodies into supermassive black holes, and from inflation*: C. Cutler and K.S. Thorne, "An Overview of Gravitational Waves Sources", in *Proceedings of the GR16 Conference on General Relativity and Gravitation*, edited by Nigel Bishop (World Scientific, 2002), in press; on our course website: Figure 4; Sections 3.3, 3.4, 2.9 and 3.6; available on course web site.
8. *Geodesic orbits of small-mass objects around supermassive black holes*
- a. For a nonspinning black hole the metric is Schwarzschild, and the orbits are discussed in all textbooks on general relativity, e.g. chapter 25 of Misner, Thorne and Wheeler, and chapter 11 of Schutz.
 - b. For a spinning black hole the metric is Kerr, and analytic equations for the orbits can be derived using Hamilton-Jacobi theory; see, e.g., Section 33.5 and Exercise 33.7 of Misner, Thorne and Wheeler. For insight into these orbits, see the following three references:
 - c. Daniel Wilkins, "Bound geodesics in the Kerr metric", *Phys. Rev. D*, **5**, 814–822 (1972). This paper focuses on non-equatorial orbits of finite-rest-mass particles, especially orbits of constant Boyer-Lindquist radial coordinate r .
 - d. Scott A. Hughes, "Nearly horizon skimming orbits of Kerr black holes", *Physical Review D*, **63**, 064016 (2001); gr-qc/0101023
 - e. James M. Bardeen, "Timelike and null geodesics in the Kerr metric", in *Black Holes*, edited by C. DeWitt and B.S. DeWitt (Gordon and Breach, 1973), pp. 213–239. This paper treats equatorial orbits, and nonequatorial photon orbits. It gives embedding diagrams for equatorial slices through the hole's exterior spacetime geometry, and it computes the shadow that a spinning hole would cast on the light from a star field behind the hole.
7. *Gravitational waveforms from small-mass objects orbiting supermassive black holes, and evolution of the orbits due to radiation reaction*
- a. For noncircular orbits in a spinning hole's equatorial plane: A near final draft of a manuscript by Costas Glampedakis and Daniel Kennefick, which is on our course's web site.

- b. For circular orbits (orbits of constant radial coordinate) inclined to the equatorial plane: The following paper by Scott A. Hughes and earlier ones referred to therein: S.A. Hughes, “Evolution of circular, non-equatorial orbits of Kerr black holes due to gravitational wave emission: II. Inspiral trajectories and gravitational waveforms”, *Phys. Rev. D*, **64**, 064004 (2001).

9. Parametric amplification of big-bang gravitational waves by inflation

- a. For the fundamental theory, along the lines of Kip’s lecture and an exercise below, see, e.g., Sections 4 and 5 of L.P. Grishchuk, “Quantum Effects in Cosmology,” *Classical and Quantum Gravity*, **10**, 2449 (1993).
- b. For the predictions of standard inflation and comparison with detection capabilities, see, e.g., M.S. turner, *Phys. Rev. D*, **55**, 435 (1997).
- c. For predictions of nonstandard cosmological models for the very early universe, some of which give much stronger waves than standard inflation, see M. Giovannini, R. Brustein, M. Gasperini and G. Veneziano, *Phys. Lett. B*, **361**, 45 (1995); M. Maggiore, *Phys. Rep.*, **331**, 283 (2000); L.P. Grishchuk, gr-qc/0002035.

Assignment, to be turned in at beginning of class on Wednesday 13 March by students registered in the course:

- A. State what reading you have done, related to the course, during this past week.
- B. Work those exercises, from the list below, that are useful for you (i.e. that are at the appropriate level for you [neither much too hard nor too easy] and that have a ratio of grunge to learning that is reasonable.
- C. If A. and B. do not constitute enough to have taught you a reasonable amount about this week’s topic, then do one or more of the following:
 - i. If you already know a lot about this week’s topic, just say so and stop.
 - ii. Invent your own exercises and work them.
 - iii. Carry out further reading and state what you have done.
 - iv. Seek private tutoring from a knowledgeable person about this week’s topic.
 - v. Pursue some other method of learning about this week’s topic, and state what you have done.

EXERCISES

Exercises related to Phinney's Lecture 15

- 1. Mass segregation in a galactic nucleus** Consider a black hole of mass M moving with speed V through a galactic nucleus consisting of a huge number N of stars. Idealize the stars as all having identical masses $m \ll M$ and denote by σ^2 their velocity dispersion (their mean square speed). Near encounters between the black hole and individual stars cause the hole's speed to change in a stochastic way. Ultimately a statistical equilibrium is reached between the hole's speed and the stars' speeds. Standard statistical-mechanical arguments tell us that in this statistical equilibrium, the mean value (over a long time) of the hole's kinetic energy is the same as the mean kinetic energies of the individual stars, $\frac{1}{2}M\langle V^2 \rangle = \frac{1}{2}m\sigma^2$. Since the hole is much more massive than the stars, this means that, in statistical equilibrium the hole's rms velocity $V_{\text{rms}} = \sigma\sqrt{m/M}$ will be much smaller than that of the individual stars.

 - a. Explain why, in statistical equilibrium, the black hole will be found near the center of the galactic nucleus.
 - b. Derive a formula for the mean radius r from the galactic center, to which the black hole wanders in this statistical equilibrium, as a function of σ^2 , M and the number density of stars n . Insert a reasonable value for n and derive a numerical value for r .
 - c. Explain why, in statistical equilibrium, there should be a "mass segregation" in the nucleus of a galaxy, with the more massive objects confined to smaller radii than the less massive objects.
- 2. Dynamical friction** Suppose that the massive black hole of the previous exercise is moving with a speed much larger than its equilibrium speed $V_{\text{rms}} = \sigma\sqrt{m/M}$. Then near encounters with individual stars, on average, must slow the hole down so that ultimately it will reach equilibrium. This slow-down is a friction-like effect and is called *dynamical friction*. In his lecture, Phinney sketched an analysis of this dynamical friction, assuming that the hole's speed V (which he called the "test-mass speed" V_t) is large compared to the stars' velocity dispersion σ . In this exercise you will do an approximate analysis for V large compared to the equilibrated speed, $\sigma\sqrt{m/M}$, including the regime $V \lesssim \sigma$, and you will learn the origin of the $\ln \lambda$ term that Phinney pulled out of the air in his lecture.

 - a. Consider, first, an encounter between the black hole and a star that initially is at rest, with the hole having an initial velocity \mathbf{V} and impact parameter b . Show that there is a critical impact parameter, $b_o = 2M/v^2$ such that, if $b \lesssim b_o$ the star's motion, as seen in the hole's rest frame, will be deflected through a large angle, but if $b \gg b_o$ it will be deflected through a small angle. Phinney treated the case of large-angle deflections in his lecture. Actually, the small-angle deflections dominate and produce the $\ln \Lambda$ term; so we shall confine attention to the small-angle case.
 - b. Compute how much energy is transferred to the star (regarded as initially at rest) by the near encounter with $b \ll b_o$.

- c. Suppose all the stars are at rest, with number density of stars n and stellar mass density $\rho_* = mn$. By integrating up the energy loss to stars with all impact parameters $b \lesssim b_o$, derive a formula for the rate of energy loss. Phinney's result, in his less careful derivation in his lecture, was

$$\frac{dE}{dt} = \frac{-G^2 M^2 \rho_* \ln \Lambda}{V}. \quad (1)$$

From your derivation, deduce what the quantity Λ is and estimate the value of $\ln \Lambda$ for a galactic nucleus. Your result should differ from Phinney's by only a multiplicative constant.

- d. From your result, give a formula for the timescale $\tau_{\text{fric}} = E(dE/dt)^{-1}$ for dynamical friction to make the hole's kinetic energy decay. Give numerical estimates for τ .
- e. If the hole's speed is large compared to the stars' velocity dispersion σ , then the above analysis, and Phinney's, are correct. Suppose, instead, that $\sigma \gg V \gg \sigma\sqrt{m/M}$. Then some of the stars with speeds larger than V will speed the hole up and some will slow it down, depending on the direction of the stars' motions; and on average these high speed stars (it turns out) will have a negligible influence compared to the ones moving more slowly than the hole. Therefore, we can estimate the timescale for tidal friction by replacing the star density $\rho_* = nm$ in the above analysis by just the density of those stars that have speeds less than the hole's speed. Statistical mechanics tells us that the stars' speeds will be given by a Maxwellian distribution,

$$\frac{dN}{dv} = \frac{n}{(2\pi\sigma^2)^{3/2}} v^2 e^{-v^2/2\sigma^2} \quad (2)$$

Use this and the above argument to deduce a simple formula for the timescale for dynamical friction when $\sigma \gg V \gg \sigma\sqrt{m/M}$. Plot the dependence of this timescale on V for the entire range from $V \gg \sigma$ to near equilibration, $V \sim \sigma\sqrt{m/M}$.

- e. For a detailed, textbook analysis of dynamical friction, see Section 7.1 of James Binney and Scott Tremaine, *Galactic Dynamics* (Princeton University Press, 1987).

- 3. Tidal friction and gravitational radiation reaction for black-hole binaries in galactic nuclei** In reference 2.a above, by Begelman, Blandford and Rees, there is a detailed discussion of the formation and evolution of black hole binaries in galactic nuclei, including, most importantly for us, the influences of tidal friction and the influences of gravitational radiation reaction. Read this reference carefully, and using your formulas for the timescale for dynamical friction, and formulas from previous weeks for the timescale for gravitational radiation reaction, elucidate the claims made by Begelman, Blandford and Rees about the binary's evolution. More specifically, explain the origin of all the *non-obvious* equations (especially timescales) in items (1) through (8) on the first page of this article. Your explanations could involve, for example, verifying the equations on the basis of your results in Exercises 1 and 2 above or on the basis of results derived earlier in this class, or on the basis of results in Phinney's lecture.

Exercises related to the Thorne's Lecture 16

4. Waveforms from a Black-Hole Binaries at Cosmological Distances. Consider a binary system at distances from earth so large that cosmological effects must be taken into account. In his lecture, Kip deduced the influence of cosmology on the observed waveform in the Newtonian, quadrupole-moment approximation. In this exercise you will deduce the influence of cosmology for the exact waveforms, including inspiral, merger and ringdown. [All parts of this exercise should be quite easy except perhaps part e, which will require some thought but not much calculation.]

- a.** The binary's evolution, is fully determined, via Einstein's field equations, by initial conditions at some fiducial time long before its merger — say $t = 10^6 M$ where M is the binary's total mass, and t is time as measured in the binary's local asymptotic rest frame (a frame close to the binary but at large enough radii that the gravitational redshift induced by the black holes is negligible). Introduce spherical polar coordinates in the local asymptotic rest frame, with the polar axis along the initial orbital angular momentum so the binary orbit is in the plane $\theta = \pi/2$. Assume that the initial orbit is elliptical with ellipticity e , and orient the coordinates so the direction ($\theta = \pi/2, \phi = 0$) is along the ellipse's major axis. Denote by M_1 and \mathbf{S}_1 the mass and spin of hole 1 at the initial time and by M_2 and \mathbf{S}_2 those of hole 2, with the spin directions defined relative to our chosen polar coordinates. Explain why the binary's initial spacetime geometry must be fully determined by the chosen initial time (i.e. by the time to merger) and the five parameters $M_1, M_2, \mathbf{S}_1, \mathbf{S}_2, e$. [Hint: appeal to the black-hole “no hair theorem” for each hole.]
- b.** As measured in the local wave zone (which is part of the binary's local asymptotic rest frame), the emitted waveforms (all the way from the initial state, through inspiral, merger and ringdown) must be functions solely of the initial parameters, the direction (θ, ϕ) to the observer, and radius r and time t ; and these waveforms must die out as $1/r$. In other words, they must have the form

$$h_J = \frac{H_J(M_1, M_2, \mathbf{S}_1, \mathbf{S}_2, e; \theta, \phi; t)}{r}, \quad (3)$$

where $J = +$ or \times and H_J is some function which numerical relativists would like to compute.

- c.** Because our universe is known to be very nearly spatially flat, we can write its cosmological spacetime metric in the form

$$ds^2 = -dt^2 + a^2(t)[d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2)]. \quad (4)$$

Here $a(t)$ is the expansion factor of the universe, which is essentially constant over the time $t = 10^6 M$ for the binary to inspiral and merge, but increases significantly from the time t_e at which the waves are emitted until the time t_o at which the waves are observed on earth. [The Einstein field equations, together with the stress-energy tensor of the fields and matter that fill the universe, determine $a(t)$; see any textbook

on cosmology or general relativity.] As the waves propagate, their wavelength gets expanded in proportion to the expansion of the universe (one can easily show), so the redshift is given by $1 + z \equiv \lambda_o/\lambda_e = a(t_o)/a(t_e)$, where z is the “cosmological redshift.” Show that the time of arrival at earth, t_o , of that bit of wave which was emitted at time t_e is given by $t_o = t_e/\zeta + \text{constant}$, where $\zeta \equiv 1 + z$ and the constant depends only on the distance from source to earth.

- d. Knowing that the waves’ amplitude dies out with distance as $1/\sqrt{\mathcal{A}}$, where \mathcal{A} is the cross sectional area of a bundle of rays, show that $h_J \propto 1/r_{\text{PM}}$ where r_{PM} (called the *proper motion distance*) is

$$r_{\text{PM}} = a(t)\chi . \quad (5)$$

(As the waves propagate radially outward along rays, χ increases, and so also does t because of the expansion of the universe as the rays travel.)

- e. Show that the waveforms measured at earth have the form

$$\frac{h_J = H_J(\zeta M_1, \zeta M_2, \zeta^2 \mathbf{S}_1, \zeta^2 \mathbf{S}_2, e; \theta, \phi; t_o + \text{constant})}{\zeta r_{\text{PM}}} . \quad (6)$$

[Hints: (i) use the fact that the waves’ phase must be conserved along rays; (ii) use the fact that h_J is dimensionless, and rewrite the waveform in the local wave zone solely in terms of dimensionless quantities.]

- f. Show, based on Eq. (6), that by comparing the observed waveform with high-precision numerical simulations of the binary’s inspiral and merger, one should be able to deduce (i) the “redshifted initial masses” $\zeta M_1, \zeta M_2$, (ii) the “redshifted initial spins” $\zeta^2 \mathbf{S}_1, \zeta^2 \mathbf{S}_2$, (iii) the initial eccentricity e , (iv) the direction (θ, ϕ) from the binary to the earth, relative to the binary’s initial orbital plane and major-axis direction, and (v) the *luminosity distance* $r_L \equiv \zeta r_{\text{PM}}$ from the earth to the binary.

- 5. Proper-Motion Distance and Luminosity Distance vs Redshift** As an aid in solving this problem, introduce a new cosmological time coordinate η defined by the solution to the differential equation $dt/a(t) = d\eta$. Then the cosmological metric (4) can be rewritten as

$$ds^2 = a^2(\eta)[-d\eta^2 + d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2)] . \quad (7)$$

Suppose, for simplicity, that the universe expands according to the law $a(\eta) \propto \eta^q$ where q is some positive number, so the big bang occurred at $\eta = 0$. Denote by η_o the value of η today and by $a_o = a(\eta_o)$ the value of the expansion factor today. Derive a formula for the proper-motion distance of a source of radiation (gravitational wave or electromagnetic) as a function of a_o, η_o and the source’s redshift $\zeta = 1 + z, r_{\text{PM}}(\zeta, a_o, \eta_o)$; and from that derive a formula for the luminosity distance r_L . Show that in the limit of very large redshifts, r_{PM} approaches a constant, while r_L increases linearly with ζ . This conclusion holds true also for more realistic (and complicated) evolutions $a(\eta)$ of the expansion factor, except in the very early (inflationary) era of the universe — an epoch in which there presumably were no discrete sources of gravitational waves that we might hope to see today.

- 6. Gravitational Waves from the Big Bang, Amplified by Inflation.** Consider gravitational waves produced in the big bang and propagating through the expanding universe. Write the

universe's spacetime metric in the forms (4) and (7) but using Cartesian coordinates rather than spherical ones, so

$$ds^2 = -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2] = a^2(\eta)[-d\eta^2 + dx^2 + dy^2 + dz^2] . \quad (8)$$

- a.** Show that in the geometric optics limit, when the waves are planar and propagate in the z direction and have only $+$ polarization, they must have the form that Kip wrote down in his lecture:

$$h_{xx} = -h_{yy} \equiv h_+ = \left(\frac{A}{a(\eta)} e^{-i\kappa\eta} \right) e^{i\kappa z} . \quad (9)$$

with all other components vanishing. Here A is a constant, which Kip denoted μ in his lecture.

- b.** What are the waves' frequency and wavelength as measured by an observer at rest (fixed (x, y, z) in the expanding universe)?
- c.** Motivated by part a, and following Kip's procedure in his lecture, write the nonzero components of the metric perturbation in the form

$$h_{xx} = -h_{yy} \equiv h_+ = \frac{\mu(\eta)}{a(\eta)} e^{i\kappa z} , \quad (10)$$

where $\mu(\eta)$ is an (as yet) unknown function that Kip stupidly denoted λ in his lecture. The Einstein field equations, linearized around the cosmological metric (8), turn out to have the form that Kip wrote down in his lecture

$$\mu_{,\eta\eta} + \left(\kappa^2 - \frac{a_{,\eta\eta}}{a} \right) \mu = 0 . \quad (11)$$

- c.** Show that this linear ordinary differential equation for $\mu(\eta)$ has the wave solution (9) whenever $\kappa^2 \gg |a_{,\eta\eta}/a|$ (as is the case today), and has the solution

$$\mu = B a + C a \int^\eta \frac{d\eta'}{a^2(\eta')} , \quad (12)$$

with B and C constants, whenever $\kappa^2 \ll |a_{,\eta\eta}/a|$. Show further that because a is an increasing function (corresponding to universal expansion), the term proportional to C dies out with time compared to that proportional to B .

- d.** During the early inflationary era of the universe, the expansion law is thought to have been $a \propto e^{Ht}$ where H is a constant. This era is thought to have been followed by a radiation-dominated era in which $a \propto t^{1/2}$; and then, after a universal age of about 100,000 years, a matter-dominated era in which $a \propto t^{2/3}$. (Observations suggest that near the universe's present epoch, the universe has returned to inflation — a conclusion not at all understood theoretically. You can ignore that possibility in this exercise.) Derive formulas for $a_{,\eta\eta}/a$ in the inflationary, radiation-dominated, and matter-dominated eras. Graph the resulting $a_{,\eta\eta}/a$ versus η , and show that it has the form Kip drew in his lecture.

- d.** Show that the solutions for $\mu(\eta)$ derived above correspond to (i) waves that propagate freely, in the geometric optics manner with $h_+ \propto e^{-i\kappa\eta}/a(\eta)$, in the exceedingly early universe, (ii) a *frozen* metric perturbation (h_+ independent of time η) during the late part of the inflationary era, and (iii) waves that propagate freely thereafter — but that might freeze again for a short while in the matter dominated era, if their wavelengths are long enough.
- e.** We know that when the waves propagate freely, in the geometric-optics manner, their gravitons are conserved; see Exercise 4 of Week 5. Show that the ratio of the number of gravitons when the metric perturbation unfreezes, at the end of inflation, to the number when it freezes in the midst of inflation, is

$$\frac{N_{\text{unfreeze}}}{N_{\text{freeze}}} \simeq \left(\frac{a_{\text{unfreeze}}}{a_{\text{freeze}}} \right)^2, \quad (13)$$

where a_{unfreeze} is the value of the expansion factor at the end of inflation, when the perturbation unfreezes and is transformed back into freely propagating waves, and a_{freeze} is the value of the expansion factor during inflation, when the initial freely propagating waves get frozen. It is presumed that the gravitational waves coming off the big bang consisted solely of vacuum fluctuations (half a graviton per mode of the graviton field). The amplification of these vacuum fluctuations corresponds, quantum mechanically, to the creation of large numbers of real gravitons during the inflationary era.

- f.** Show that there is a gravitational-wave frequency today above which the big-bang waves should not have frozen during inflation, and therefore above which there should not have been any amplification of the number of gravitons. That frequency is estimated to be very roughly 10^9 Hz, so the amplification should have been very great in all the frequency bands in which gravitational waves are currently being sought.

7. Geodesic Orbits Around a Nonspinning Black Hole.

- a.** Consider a small-mass object moving on a geodesic orbit in the curved spacetime around a nonspinning supermassive black hole. The hole's spacetime metric is Schwarzschild

$$ds^2 = -(1 - 2M/r)dt^2 + \frac{dr^2}{1 - 2M/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (14)$$

where $M \gg \mu$ is the hole's mass and μ is the orbiting object's mass. Orient the coordinate system so the orbit is in the “equatorial plane” $\theta = \pi/2$. Then because the metric is independent of t and ϕ , the components $p_t \equiv -E$ and $p_\phi \equiv L$ of the object's 4-momentum are conserved along the orbit. Explain why E should be interpreted as the particle's energy and L as its angular momentum. Show that the particle's angular and radial motion are governed by the equations

$$\frac{d\phi}{d\tau} = \frac{\tilde{L}}{r^2}, \quad \left(\frac{dr}{d\tau} \right)^2 + \tilde{V}(r) = \tilde{E}^2 \quad (15)$$

where the effective radial potential is given by

$$\tilde{V}(r) = (1 - 2M/r)(1 + \tilde{L}^2/r^2), \quad (16)$$

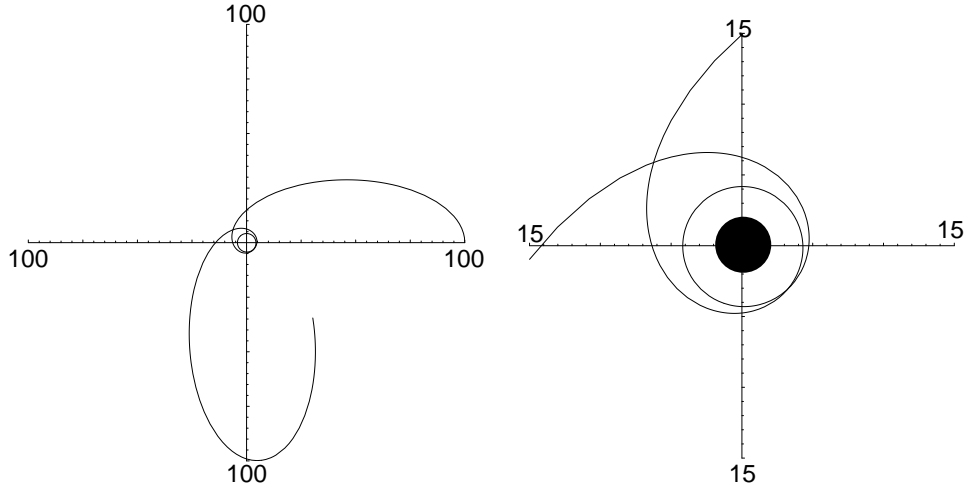


Figure 1:

with $\tilde{L} \equiv L/\mu$ and $\tilde{E} \equiv E/\mu$ the angular momentum and energy per unit mass. (Note: the notation is slightly different from that which Kip used in his lecture.)

- b.** The equations of motion (15), (16) have the following analytic solution:

$$u = u_1 + (u_2 - u_1) \text{sn}^2 \left(\phi \sqrt{\frac{(u_3 - u_1)}{2}} \mid \frac{u_2 - u_1}{u_3 - u_1} \right), \quad (17)$$

where $u = M/r$; the u_j are related by $u_1 + u_2 + u_3 = 1/2$, and $u_1 < u_2 < u_3$; and $\text{sn}(\theta|m)$ is the Jacobi Elliptic function with modulus m (sometimes denoted $m = k^2$). Show that the orbit's apastron and periastron are at $r_1 = M/u_1$ and $r_2 = M/u_2$.

- c.** Use Mathematica or Maple or Macsyma or some other graphics tool to plot the shapes of these orbits in spherical polar coordinates (r, ϕ) for various values of the apastron and periastron. Explore, graphically, what happens as the apastron gets closer and closer to the black hole (so the orbit becomes more and more relativistic). As an example, for $r_1 = 100M$ and $r_2 = 4.166M$, your plot should look like Fig. 1.
- d.** Explain your orbit shapes using the effective potential (16).