

**WEEK 8: COMPACT BINARIES AS GW SOURCES**  
**Lectures 13 by Phinney, and 14 by Thorne & Buonanno**

**Recommended Reading:**

As last week, there are no textbook treatments of the material covered in this week's lectures. The following materials may be useful. There is far more material here than any one person should try to read. Choose a reasonable amount, in the area of greatest interest or relevance to you, and read or browse it.

1. Phinney's handout showing
  - a. Specific compact binaries in our galaxy
  - b. The evolution of the radius and core mass of each of three nonrotating stars that are not in binaries: stars with masses  $15M_{\odot}$ ,  $5M_{\odot}$ , and  $1.5M_{\odot}$ . These give insights into some of the phenomena (evolution to red giant phase, core formation in process of red giant formation, Hertzsprung gap, ...) that influence binary evolution.
2. K. Belczynski, V. Kalogera, and T. Bulik, "A Comprehensive Study of Binary Compact Objects as Gravitational Wave Sources: Evolutionary Channels, Rates, and Physical Properties", *Astrophys. J.*, in press; available at <http://lanl.arXiv.org/abs/astro-ph/0111452>. This paper presents the details of a population synthesis calculation (Monte Carlo evolution of more than  $10^6$  binaries), which focuses on the rate of NS and BH binary mergers for LIGO. Among the evolutionary processes discussed are those in Phinney's lecture — though in some cases from a different viewpoint. There are a number of similar population synthesis papers by other researchers (see references in the first paragraph on page 2); this paper has the advantages of containing a review of single star evolution followed by the discussion of binary evolution, and of being the most recent paper on this subject. This paper is quite long, so you are encouraged to browse it, reading carefully only the most enlightening parts.
3. C. Cutler and K.S. Thorne, "An Overview of Gravitational Waves Sources", in *Proceedings of the GR16 Conference on General Relativity and Gravitation*, edited by Nigel Bishop (World Scientific, 2002), in press. Cutler and Thorne just finished writing this review article a few days ago and it is not yet on the Los Alamos data base, but it *is* on our course web site. For this week the relevant portions are those that summarize signal strengths, event rates, and information carried for waves from compact binaries, in particular:
  - a. Section 2.1 on LIGO noise curves.
  - b. Those portions of Section 2.2 and Box 1 that deal with NS and BH binaries.
  - c. Section 2.3 on NS and BH binary inspiral as seen by LIGO.
  - d. Section 3.1 on the LISA noise curve.

- e. Section 3.2 on binaries as observed by LISA.
4. T. Damour, B.R. Iyer and B.S. Sathyaprakash, “A Comparison of Search Templates for Gravitational Waves from Binary Inspiral,” *Phys. Rev. D*, **63**, 044023 (2001). This article is available at <http://xxx.lanl.gov/abs/gr-qc/0010009> and also on the Physical Review web site. It covers much of the material in Buonanno’s lecture, though restricted to binaries with vanishing spins and with a lot of extraneous detail that you may wish to ignore. The portions of the paper that correspond most closely to Buonanno’s lecture are
    - a. The introduction to Section II, and Sec. II.A, on the post-Newtonian-based Taylor expansion of the binary’s phase; and Tables I and II, which give the coefficients that appear in the Taylor expansion.
    - b. Section III on the frequency-domain expansion of the phase.
    - c. Sections IIB and IV on accelerating the convergence of the expansion by using Padé approximants and other techniques. You may want to just browse this material.

**Possible Supplementary Reading:**

5. S. Portegies Zwart and S.L.W. McMillan, “Black Hole Mergers in the Universe”, *Astrophys. J.*, **528**, L17 (2000); available at <http://xxx.lanl.gov/abs/astro-ph/9910061> . This paper discusses the production of black hole binaries in dense star clusters, where the evolutionary processes (involving captures of non-binary holes by each other) are very different than in normal regions of a galaxy (in the “field”).
6. M.C. Miller and D.P. Hamilton, “Four-Body Effects in Globular Cluster Black Hole Coalescence”, *Monthly Notices of the Royal Astronomical Society*, **330**, 232 (2002); available at <http://lanl.arXiv.org/abs/astro-ph/0202298> . This extends the results of the above paper, showing that in some dense star clusters a massive black hole may grow by successive captures of smaller black holes. This may have important consequences for LIGO, but they have not yet been explored quantitatively.
7. D.R. Lorimer, “Binary and Millisecond Pulsars at the New Millenium,” *Living Reviews in Relativity*, **4**, 2001-5; available at <http://www.livingreviews.org/Articles/Volume4/2001-5lorimer/>
8. Either of the following two articles on spin-orbit and spin-spin coupling in NS and BH binaries, most especially the precessions of the orbit and spins and the resulting modulations of the waves. These are best accessed from the Physical Review Web Site rather than the Los Alamos data base — Apostolatos et al is not on Los Alamos, and Kidder is on in a double spaced, large-type form that eats up a huge number of pages.
  - a. T.A. Apostolatos, C. Cutler, G.J. sussman, and K.S. Thorne, “Spin-induced orbital precession and its modulation of the gravitational waveforms from merging binaries”, *Phys. Rev. D*, **49**, 6274 (1994); available at <http://prola.aps.org/> .

- b. L. Kidder, “Coalescing binary systems of compact objects to Post<sup>5/2</sup>-Newtonian order.” *Phys. Rev. D*, **52**, 821 (1005); available at <http://prola.aps.org/> . This gives the equations and calculations to higher order in the orbital velocity  $v$  than in the Apostolatos paper.
9. T. Damour, B.R. Iyer and B.S. Sathyaprakash, “Improved filters for gravitational waves from inspiraling compact binaries”, *Phys. Rev. D* **57**, 885 (1998); <http://xxx.lanl.gov/abs/gr-qc/9708034>. This paper introduces the technique of Pade approximants for accelerating the convergence of the post-Newtonian expansion.
  10. A. Buonanno and T. Damour, “Effective one-body approach to general relativistic two-body dynamics”, *Phys. Rev. D* **59**, 084006; <http://xxx.lanl.gov/abs/gr-qc/9811091>. This paper introduces the effective one-body approach to circumventing the problem of slow convergence of the post-Newtonian expansion.

**Assignment, to be turned in at beginning of class on Wednesday 6 March by students registered in the course:**

- A. State what reading you have done, related to the course, during this past week.
- B. Work those exercises, from the list below, that are useful for you (i.e. that are at the appropriate level for you [neither much too hard nor too easy] and that have a ratio of grunge to learning that is reasonable.
- C. If A. and B. do not constitute enough to have taught you a reasonable amount about this week’s topic, then do one or more of the following:
  - i. If you already know a lot about this week’s topic, just say so and stop.
  - ii. Invent your own exercises and work them.
  - iii. Carry out further reading and state what you have done.
  - iv. Seek private tutoring from a knowledgeable person about this week’s topic.
  - v. Pursue some other method of learning about this week’s topic, and state what you have done.

## EXERCISES

### Exercise related to Phinney’s Lecture 13

1. **Examples of Binary Evolution** [courtesy Sterl Phinney]. *Note: If some of the terminology in this problem is alien to you, look it up in an astrophysics textbook or ask an astrophysics student or professor to explain it.* See if you can understand approximately the numerical values in the theoretically computed binary evolution shown in Table 1, which begins with

Table 1: An example of binary evolution. Notation: MS=main sequence; HG=Hertzsprung gap; cHeB=core Helium burning; nHEM=naked Helium main sequence; nHeH=naked Helium star, Hertzsprung gap; NS=neutron star; BH=black hole; RL1 = Roche lobe of star 1; RL2 = Roche lobe of star 2.

Time, Myr	$M_1/M_\odot$	$M_2/M_\odot$	Separation/ $R_\odot$	$e$	What's Happening
0.0000	13.600 MS	8.000 MS	100.168	0.00	zero-age main sequence star
14.7175	13.341 HG	8.004 MS	101.277	0.00	
14.7418	13.338 HG	8.004 MS	101.329	0.00	$M_1$ fills RL1
14.7497	13.330 cHeB	8.010 MS	101.279	0.00	
14.7841	3.284 cHeB	18.051 MS	298.168	0.00	end of mass transfer
14.8029	3.262 nHeM	18.067 MS	297.662	0.00	
16.8681	2.978 nHeH	17.997 MS	302.713	0.00	supernova
17.0546	1.346 NS	17.993 MS	326.675	0.07	
22.9777	1.346 NS	17.533 HG	334.431	0.07	
22.9963	1.346 NS	17.521 cHeB	334.674	0.07	
23.0150	1.346 NS	17.491 cHeB	335.343	0.07	$M_2$ fills Roche lobe
23.0150	1.346 NS	4.711 nHeM	1.697	0.00	end common envelope spiral in
*23.5695	1.353 NS	4.470 nHeM	1.296	0.00	$M_2$ fills Roche lobe again
*24.3720	1.383 NS	3.927 nHeH	0.932	0.00	supernova
24.3720	1.383 NS	1.371 NS	2.125	0.00	0.000 0.000
653.4350	1.383 NS	1.371 NS	0.000	0.00	$M_1, M_2$ fill RL1, RL2
653.4350	2.754 BH	0.000	0.000	0.00	coalescence of NS+NS

Table 2: Another example of binary evolution. Notation is same as in Table 1.

Time, Myr	$M_1/M_\odot$	$M_2/M_\odot$	Separation/ $R_\odot$	$e$	What's Happening
0.0000	13.000 MS	10.000 MS	102.287	0.00	ZAMS start
15.7681	12.773 HG	9.960 MS	103.370	0.00	
15.7821	12.772 HG	9.960 MS	103.398	0.00	$M_1$ fills RL1
15.8036	3.040 nHeM	12.741 MS	73.167	0.00	end of transfer
16.1395	3.016 nHeM	12.738 MS	73.296	0.00	
18.1694	2.796 nHeH	12.713 MS	74.434	0.00	supernova
18.3771	1.333 NS	12.722 MS	82.218	0.09	
23.8042	1.333 NS	12.596 HG	82.910	0.09	
23.8204	1.333 NS	12.595 HG	83.009	0.09	$M_2$ fills RL2
23.8204					NS spirals into core of giant: Thorne-Zytkow object formed.

13.6 $M_\odot$  and 8 $M_\odot$  stars in an initially circular orbit with period 25 days. This is the sort of model popular with some researchers for the formation of double neutron star systems like the observed PSR 1913+16 and PSR 1534+12. The one unrealistic feature of the evolution shown below is that when the neutron stars are born in supernovae, they are given no kicks (vanishing “natal kick velocity”); in more realistic evolutions there is a significant natal kick velocity.

- a. From the graphs and principles given in Phinney’s lecture, figure out approximately the following, for yourself: (i) the times and Roche lobe radii of the mass-transfer events. (ii) whether those transfers are dynamically or thermally unstable, and the timescale on which they ought to occur, (iii) the changes in the orbital separations due to mass transfer.
- b. Also ask yourself: if the neutron stars *were* given natal kicks, how large could they be without: (i) the first supernova being likely to unbind the binary, and (ii) the second supernova being likely to unbind the binary. Do either of these explain why NS-NS binaries are rare? Which is most important?
- c. In this computed evolution, the accretion rate onto the first-born neutron star was Eddington limited. Estimate the Bondi accretion rate (the rate of accretion if there were no outward force of radiation on the infalling electrons) during the stages indicated by \*. What could have been the final state if the accretion rate were the Bondi rate?
- c. At one point in the evolution, there appears to be a bug in the algorithms the code used. Can you find it?
- d. With the same evolution code, a binary with the same initial orbital period of 25 days, but now with slightly different initial stellar masses — 13 $M_\odot$  (vs 13.6) and 10 $M_\odot$  (vs 8) — leads to the quite different evolution shown in Table 2. See if you can understand the sensitivity to the initial conditions and the cause of the difference in final state.

## Exercise related to the Thorne/Buonanno Lecture 14

- 2. Post-Newtonian Expansion of Waveform from an Inspiring, Circular Binary with an Extreme Mass Ratio.** *Note: This exercise looks very long; actually it is less long than it looks — you are led by the hand through calculations that, in most cases, are rather easy and quick.* Consider a binary consisting of a heavy black hole with mass  $M$  orbited by a neutron star with mass  $\mu \ll M$ , and assume that the spins of the hole and the star are negligible. The black hole's spacetime metric is given by Schwarzschild's formula

$$ds^2 = -(1 - 2M/r)dt^2 + \frac{dr^2}{1 - 2M/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1)$$

The neutron star moves in a circular geodesic orbit in the equatorial plane  $\theta = \pi/2$ . The components of the star's 4-momentum are  $p^\alpha = mdx^\alpha/d\tau$ , where  $\tau$  is proper time along its orbit.

- a.** Explain why the orbital angular velocity, as measured by an observer far from the binary, is  $\Omega = d\phi/dt = p^\phi/p^t$ . One can show that the usual Keplerian formula

$$\Omega = \sqrt{M/r^3} \quad (2)$$

(with  $r$  the radius of the orbit) is valid without change (valid fully relativistically) for this (and any) circular geodesic orbit in the Schwarzschild metric; see, e.g., Eq. (11.21) of Schutz, *A First Course in General Relativity* or Exercise 25.19 of Misner, Thorne and Wheeler *Gravitation*.

- b.** Because the Schwarzschild metric is independent of the time coordinate, the covariant component of the 4-momentum,  $p_t \equiv -E$ , is a constant of the motion not just for circular geodesic orbits but for any geodesic orbit (see Exercise 6 in Week 5). The quantity  $E$  is the conserved energy of the body that moves along the orbit. For the neutron star's circular orbit, use the relations  $p^\alpha p^\beta g_{\alpha\beta} = -\mu^2$  (explain where this comes from) and  $p^\phi/p^t = \Omega = \sqrt{M/r^2}$  to show that

$$E = \mu \frac{1 - 2M/r}{\sqrt{1 - 3M/r}}. \quad (3)$$

This is an exact relation, not an approximate, post-Newtonian one; but we shall take its post-Newtonian limit below.

- c.** Note that as  $r \rightarrow \infty$ ,  $E \rightarrow \mu$ . This means that  $E$  contains the star's rest-mass energy. Show that at large radii,  $E \simeq \mu - \mu M/2r$ . This is the standard Newtonian formula for the orbital energy: rest mass  $\mu$  plus kinetic energy equal to  $\mu M/2r$  plus gravitational potential energy  $-\mu M/r$ .
- d.** Draw a graph of  $E(r)$ . Notice that it decreases monotonically with decreasing  $r$  until  $r = 6M$ , where it begins increasing. As energy is gradually lost to gravitational waves, the radius will shrink from  $r \gg 6M$  to  $r = 6M$ . Thereafter, further losses of energy

cannot be accommodated by circular geodesic orbits. There are no such orbits with energies smaller than that at  $r = 6M$ . But energy continues to be lost to gravitational waves. What must happen (and does happen) is that the star plunges toward the hole's horizon, on a noncircular orbit, once it reaches  $r = 6M$ . Thus,  $r = 6M$  is the innermost stable circular orbit, *isco*, which Buonanno discussed in her lecture.

- e.** The orbiting neutron star emits gravitational waves that are predominantly at twice the orbital frequency,  $f = 2\Omega/2\pi = \Omega/\pi$ , though there are also harmonics at other multiples of  $\Omega/\pi$ . For simplicity we shall focus on those waves that come out at this predominant frequency  $f$ . Explain why  $f$  is the frequency measured by an observer far from the hole, but not the frequency measured near the neutron star.
- f.** Define the parameter  $v \equiv (\pi M f)^{1/3}$ . Show that for a circular geodesic orbit at any radius  $r$ ,  $v = \sqrt{M/r}$  is an exact relation. Show that at large radii  $r$ , this  $v$  is the speed of the star in its orbit. At small radii it is of order that speed, but the exact value of the speed depends on the reference frame of the measurer. Suppose that the measurer is at rest outside the black hole ( $r, \theta, \phi$  constant) at a location through which the star's orbit passes. Show that the speed the observer measures as the star whizzes by is  $v/\sqrt{1 - 2M/r}$ .
- g.** When one uses post-Newtonian techniques to compute the energy carried off by the gravitational waves (the waves' luminosity), one obtains the following formula:

$$\mathcal{F} = \frac{32}{5} \left( \frac{\mu}{M} \right)^2 v^{10} \left[ 1 - \frac{1247}{336} v^2 + 4\pi v^3 + \mathcal{O}(v^4) \right] \quad (4)$$

Verify that the term preceding the square brackets is the prediction of the quadrupole formula when the orbit is regarded as Newtonian, as derived in Exercise 4 of Week 6. The term  $(1247/336)v^2$  is a post-Newtonian correction that includes mass octupole radiation and a variety of other post-Newtonian effects. The post<sup>1.5</sup>-Newtonian term  $4\pi v^3$  is produced by the waves' tails — i.e., by that part of the waves that scatters off the black hole's spacetime geometry as it tries to escape from the hole's vicinity, propagates back in toward the hole, then deflects around the hole and reemerges, delayed relative to the prompt waves that carry the Newtonian and post-Newtonian energy. We shall be interested in studying the detectability of this tail contribution to the waves' luminosity.

- h.** Perform a post-Newtonian expansion of the orbit's energy  $E$  to obtain, up to errors of post<sup>2</sup>-Newtonian order,

$$E = \mu - \frac{1}{2} \mu v^2 \left[ 1 - \frac{3}{4} v^2 + \mathcal{O}(v^4) \right] \quad (5)$$

- i.** Show (repeating a derivation that Buonanno gave in her lecture) that the law of energy conservation,  $dE/dt = -\mathcal{F}$  implies the waves' frequency  $f$ , or equivalently  $v = (\pi M f)^{1/3}$ , evolves with time  $t$  (time as measured by observers far from the hole) in the following manner:

$$t(v) = t_{\text{ref}} + M \int_v^{v_{\text{ref}}} \frac{dE(v')/dv'}{\mathcal{F}(v')} dv'. \quad (6)$$

Here  $v_f \equiv (\pi M f_{\text{ref}})^{1/3}$  is the value of  $v$  when some reference frequency (e.g., 100 Hz, or 1000 Hz, or whatever you wish) is reached, and  $t_{\text{ref}}$  is the time at which that reference frequency is reached. Equation (6) can be thought of as giving the time  $t_f$  at which frequency  $f$ , corresponding to  $v = (\pi M f)^{1/3}$  is reached. *Derive a formula for  $t_f$  as a power series in  $v$  up through post<sup>1.5</sup>-Newtonian order.* Show that at the leading, Newtonian order your result can be expressed in terms of the binary's chirp mass  $\mathcal{M} \equiv \mu^{3/5} M^{2/5}$  and is in accord with the results of Exercise 4 of Week 6. Notice that the post-Newtonian and higher-order corrections carry information about the hole's mass  $M$ . Therefore, if the waves' frequency evolution were measured, from the Newtonian order result we could infer the chirp mass and then from the higher order corrections we could infer the hole mass  $M$ , and knowing  $\mathcal{M}$  and  $M$  we could infer the neutron star mass  $\mu$ . We could invert your expansion for  $t_f$  to get the frequency  $f$  as a power series in time  $t$ , if we wished; but we shall not need it below. Rather, in our final result at the end of this problem, we shall need time as a function of frequency,  $t_f$ .

- j.** The waves' phase  $\phi = \int 2\pi f dt$  can be thought of equally well as a function of time  $t$ , or a function of the frequency  $f$  that is reached at time  $t$ , or as a function of  $v = (\pi M f)^{1/3}$ . Show that

$$\phi(f) = \phi_{\text{ref}} + 2 \int_v^{v_{\text{ref}}} \frac{v'^3 dE(v')/dv'}{\mathcal{F}(v')} dv', \quad (7)$$

where  $\phi_{\text{ref}}$  is the value of the phase when the reference frequency is reached, and the  $v$ 's on the right hand side are to be thought of as functions of  $f$ ,  $v = (\pi M f)^{1/3}$ .

- k.** *Derive a formula for  $\phi(f)$  as a post-Newtonian expansion in  $v$ , accurate up through post<sup>1.5</sup>-Newtonian order.*
- l.** The post<sup>1.5</sup>-Newtonian term in  $\phi(f)$  is the one that arises from the tails of the waves. How many radians of phase does this term contribute, in the LIGO-II frequency band (from about 10 Hz to about 1000 Hz) in the case of a  $1.4M_{\odot}$  neutron star spiraling into a  $10M_{\odot}$  black hole? With what accuracy, roughly, would you expect that the influence of the waves' tails can be measured?
- m.** The gravitational waves measured at Earth will have the form

$$h(t) = A(t) \cos \phi(t), \quad (8)$$

where  $\phi(t)$  is the phase computed above, regarded as a function of time  $t$ , and where (see Exercise 5 in Week 5) the amplitude  $A(t)$  is  $A(t) \propto f^{2/3} \propto v^2$ , with  $f$  and  $v$  the values reached at time  $t$ . This expression for the amplitude is actually just the Newtonian order term in a post-Newtonian expansion. Since the data analysis is highly sensitive to the waves' phase evolution  $\phi(t)$  but not very sensitive to the amplitude evolution, we evaluate  $A$  only at leading, Newtonian order while evaluating  $\phi$  to as high an order as our fortitude permits. As we shall see next term, gravitational wave signal processing is best analyzed using not  $h(t)$  but instead its Fourier transform  $\tilde{h}(f) = \int_{-\infty}^{+\infty} e^{2\pi i f t} h(t) dt$ . Show that  $\tilde{h}(-f) = \tilde{h}^*(f)$  where the star denotes complex conjugation. This permits us to restrict attention to positive frequencies. The remainder of this exercise evaluates  $\tilde{h}(f)$  in preparation for the discussion of gravitational-wave data analysis next term.

n. By evaluating the Fourier transform using the stationary phase approximation, show that

$$\tilde{h}(f) = B(f)e^{i\psi_f - i\pi/4} \quad (9)$$

where the amplitude of the Fourier transform is

$$B(f) \simeq \frac{1}{2} \frac{dt_f}{df} A(t_f) \propto f^{-7/6}, \quad (10)$$

and the phase of the Fourier transform, expressed as a function of frequency  $f$ , is

$$\psi_f = 2\pi f t_f - \phi(f). \quad (11)$$

l. Use your post-Newtonian expansions for  $t_f$  and  $\phi(f)$  to obtain the following expansion for the waves' phase

$$\psi_f = 2\pi f t_{\text{ref}} - \phi_{\text{ref}} + \frac{3}{128} (\pi \mathcal{M} f)^{-5/3} \left[ 1 + \frac{3715}{756} v^{2/3} - 16\pi v + \mathcal{O}(v^{4/3}) \right]. \quad (12)$$

We shall use Eqs. (9)–(12) next term when we discuss searching for these binary waves in LIGO's noisy data.