WEEK 6: PROPAGATION OF GWs THROUGH MATTER, and SLOW-MOTION APPROXIMATION FOR WAVE GENERATION

Lectures 10 and 11

Recommended Reading:

Some of the reading this week comes from chapter 5 of the 1989 unpublished manuscript of a book by Kip: *Gravitational Waves: A New Window onto the Universe*. This book was originally typeset using ancient Unix software called "troff"; it was converted into TeX in the mid 1990s, but the conversion has not been checked carefully, so there may be a number of typos. This chapter is on our course website. Below it is cited as *K.S. Thorne, GWs: New Window*.

Other reading comes from K. S. Thorne, "The Theory of Gravitational Radiation: An Introductory Review", in *Gravitational Radiation*, eds. N. Deruelle and T. Piran (North Holland, Amsterdam, 1983), pp. 1–57. This is also on the course web site, and we have used it in earlier weeks of the course. Below it is cited as *K.S. Thorne, GWs: Introductory Review*.

1. Propagation of gravitational waves through matter:
   a. Section 5.E of chapter 5 of K. S. Thorne, *GWs: New Window*. This 1989 presentation is more complicated than the one Kip gave in Lecture 10: It treats the wave-induced perturbations of the matter's stress-energy tensor with much mathematical carefulness; the tools for this are defined in the first part of Sec. 5.B [up through Eq. (5.11)]. Kip no longer believes this carefulness is needed. The approach he took in his lecture, he thinks, is quite satisfactory, and it gives all the same answers.
   b. Section 2.4.3 of K.S. Thorne, *GWs: Introductory Review*.
2. Slow-motion approximation for GW generation: K.S. Thorne, GW's: Introductory Review: Sections 1.3, 3.1.1, 3.1.2, 3.2, and 3.3.2. Note: some of this material takes a more advanced and sophisticated viewpoint than Kip developed in his lecture.

Possible Supplementary Reading:

3. Concerning the fact that matter in the very early universe is transparent to gravitational waves all the way back to the Planck era when space and time were being created: Section 7.2 of Yakov Borisovich Zel'dovich and Igor Dimitrievich Novikov, *The Structure and Evolution of the Universe* (University of Chicago Press, 1983).
4. For an analysis of GW propation through fluids and other kinds of matter: Leonid P. Grischuk and Alexander G. Polnarev, in *General Relativity and Gravitation*, vol. 2, edited by Alan Held (Plenum, New York), pp. 393ff.
5. For multipole-moment expansions of the general relativistic gravitational field of an isolated source (including the decomponsition of its gravitational waves): Kip S. Thorne, "Multipole expansions of gravitational radiation", *Reviews of Modern Physics* **52**, 299–339 (1980). Most especially:

a. spherical harmonics in the language of symmetric, trace-free (STF) tensors: Sec. II.
b. regions of spacetime around an isolated source: Sec. III.
c. Derivation of the general outgoing-wave solution to the linearized Einstein equations in Lorenz gauge: Sec. VIII. The quadrupolar parts of this solution are the ones that Kip discussed in Lecture 11 and used to relate the radiation field to the weak-field near-zone quadrupole moments.

Assignment, to be turned in at beginning of class on Wednesday 20 February by students registered in the course:

A. State what reading you have done, related to the course, during this past week.
B. Work those exercises, from the list below, that are useful for you (i.e. that are at the appropriate level for you [neither much too hard nor too easy] and that have a ratio of grunge to learning that is reasonable.
C. If A. and B. do not constitute enough to have taught you a reasonable amount about this week's topic, then do one or more of the following:
   i. If you already know a lot about this week's topic, just say so and stop.
   ii. Invent your own exercises and work them.
   iii. Carry out further reading and state what you have done.
   iv. Seek private tutoring from a knowledgable person about this week's topic.
   v. Pursue some other method of learning about this week's topic, and state what you have done.

EXERCISES

Note: There are more exercises here than any single person is expected to work. Work only those exercises that are useful for you!

Exercises filling in the gaps in Kip's lectures

1. **GW Propagation through a Universe Filled with Neutron Stars**
   Consider the model universe filled with identical neutron stars, through which a plane, monochromatic gravitational wave (with angular frequency ω) propagates, as discussed in Kip's Lecture 10. This is a special case of the universe filled with quadrupolar oscillators discussed in Sec. 5.E.d of *GWs: New Window* (Ref. 1a of the recommended reading), and discussed in Sec. 2.4.3 of *GWs: Introductory Review* (Ref. 1b).
   a. When the waves interact with one of the neutron stars, they induce oscillations of one of the star's quadrupolar normal modes, which has angular frequency ω_o and damping time τ_*. Describe the internal motions of the neutron-star matter than you expect for these quadrupolar oscillations. Explain why you expect the star's quadrupole moment tensor $\mathcal{I}_{jk}$ to obey an oscillation equation with the form given in Eq. (2.39a) of *GWs: Introductory Review*, with the factor 1/5 replaced by a dimensionless number α of order unity.
   b Explain why the wave equation for the waves has the form (2.39b).

c. Derive the dispersion relation (2.39c) for these waves.

d. Derive Eq. (2.40) for the lengthscale ℓ over which there is significant absorption or dispersion.

e. In the real universe each galaxy such as our own contains roughly 10^9 neutron stars, since the birth rate of neutron stars is about one every 30 years and our galaxy is about 10^{10} years old. Estimate the mean density n of neutron stars in the entire universe. The neutron stars' periods of quadrupolar oscillation are of order 0.1 to 1 milliseconds, and their damping times are of order seconds to minutes — with the damping possibly due largely to emitting gravitational waves. The greatest possible absorption and dispersion of GW's will arise if the neutron stars all have identically the same eigenfrequency ω_o and the waves drive them precisely on resonance, $\omega = \omega_o$ (a highly implausible situation). In this most extreme scenario, what is the lengtscale ℓ for substantial absorption or dispersion, in light years?

2. **Near-Zone and Wave-Zone Limits of General Quadrupolar Outgoing Wave in Linearized Theory**

Consider the general quadrupolar, outgoing gravitational-wave solution to Einstein's equations in Lorenz gauge. This solution is given by the following formula, which Kip discussed in Lecture 10:

$$\bar{h}^{00} = +\left[\frac{2}{r}\mathcal{I}_{jk}(t-r)\right]_{,jk} , \quad \bar{h}^{0j} = -\left[\frac{2}{r}\dot{\mathcal{I}}_{jk}(t-r)\right]_{,k} , \quad \bar{h}^{0j} = +\left[\frac{2}{r}\ddot{\mathcal{I}}_{jk}(t-r)\right] ; \tag{1}$$

here the spatial coordinates are assumed to be Cartesian, $r = \sqrt{\delta_{jk}x^j x^k}$ is radius measured from the center of mass of the source, the dots denote time derivatives, and $\mathcal{I}_{jk}$ is the source's mass quadrupole moment. Because the basis is Cartesian, it does not matter whether spatial indices are placed up or down. Note: Eq. (1) is the mass quadrupole term in Eqs. (3.23) of *GWs: Introductory Review*.

a. Show that this trace-reversed metric perturbation satisfies the Lorenz gauge condition $\overline{h}^{\alpha\beta}{}_{,\beta} = 0$.

b. Show that it satisfies the vacuum Einstein field equation $\overline{h}_{\alpha\beta,\mu}{}^{\mu} = 0$.

c. Show that in the source's weak-field near zone this solution reduces, at leading order, to

$$\bar{h}^{00} = 6\frac{\mathcal{I}_{jk}n^j n^k}{r^3} , \quad \bar{h}^{0j} = \bar{h}^{jk} = 0 , \tag{2}$$

d. where $n^j = x^j/r$ is the unit radial vector. This expression for $\bar{h}^{00}$ is -4 times the quadrupolar part of the source's Newtonian gravitational potential. Show that quite generally, for a nearly Newtonian source in the weak-field near zone, $\bar{h}^{00}$ should be -4 times the Newtonian potential.

e Show that in the source's local wave zone the solution (1) entails a gravitational wave field given by

$$h^{\rm TT}_{jk} = \left[\frac{2}{r}\ddot{\mathcal{I}}_{jk}(t-r)\right]^{\rm TT} . \tag{3}$$

3. **Energy Carried Away by Quadrupolar Gravitational Waves**

a. Show that in the source's local wave zone, the energy flux $T^{0j}_{\rm GW}$ carried by the mass-quadrupolar gravitational wave (3) is radial in direction and is given by

$$T^{0r}_{\rm GW} = \frac{1}{8\pi r^2}\left\langle \frac{\partial^3}{\partial t^3}\mathcal{I}^{\rm TT}_{jk}(t-r)\frac{\partial^3}{\partial t^3}\mathcal{I}^{\rm TT}_{jk}(t-r)\right\rangle . \tag{4}$$

b. Show that the quantity

$$P_{jk} \equiv \delta_{JK} - n_j n_k \tag{5}$$

has the property that for any vector $\mathbf{A}$, $P_{jk}A_k$ is the projection of $\mathbf{A}$ into the plane perpendicular to the radial direction. In quantum mechanics one defines a projection operator $\hat{P}$ to be any operator that satisfies $\hat{P}\hat{P} = \hat{P}$. Show that P_{jk} is a projection operator in this quantum mechanical sense.

c. Show that the TT part of $\mathcal{I}_{jk}$ is given by

$$\mathcal{I}^{\rm TT}_{jk} = P_{ja}I_{ab}P_{bk} - P_{jk}(P_{ab}\mathcal{I}_{ab}) . \tag{6}$$

d. Using the projection operator, show that the radial flux of GW energy is

$$T^{0r}_{\rm GW} = \frac{1}{8\pi r^2}\left\langle \mathcal{I}_{jk,000}\mathcal{I}_{jk,000} - 2n_i\mathcal{I}_{ij,000}\mathcal{I}_{jk,000}n_k + \frac{1}{2}(n_j\mathcal{I}_{jk,000}n_k)^2 .\right\rangle \tag{7}$$

e. Show that, when one integrates over solid angle $d\Omega$ on a sphere surrounding the source, the following relations are true:

$$\frac{1}{4\pi}\int n_i d\Omega = 0 , \quad \frac{1}{4\pi}\int n_i n_j d\Omega = \frac{1}{3}\delta_{ij} , \quad \frac{1}{4\pi}\int n_i n_j n_k d\Omega = 0 ,$$

$$\frac{1}{4\pi}\int n_i n_j n_k n_l = \frac{1}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}+) . \tag{8}$$

[Hint: Use antisymmetry of the integrand when an odd number of $n's$ is present. For an even number of n's, argue that the answer can only involve delta functions or products of delta functions (the delta function being the metric of 3-space); write down the form it must take with a single unknown numerical coefficient; then evaluate that coefficient by considering the component with $i = j = z$ or $i = j = k = l = z$ and using $n_z = \cos\theta$ where θ is the polar angle.]

f. Explain why the wave emission produces a reduction in the source's total mass-energy given by

$$\frac{dE}{dt} = -\int T^{0r}_{\rm GW} d\Omega \tag{9}$$

where the integral is over a sphere in the local wave zone, surrounding the source.

g. Combine the equations derived above to deduce that the energy loss rate is

$$\frac{dE}{dt} = -\frac{1}{5}\left\langle \frac{\partial^3}{\partial t^3}\mathcal{I}^{\rm TT}_{jk}(t)\frac{\partial^3}{\partial t^3}\mathcal{I}^{\rm TT}_{jk}(t)\right\rangle . \tag{10}$$

h. Notice that the quadrupole moments on the right side are evaluated at the same time t (averaged over several periods of the waves) as the time t at which dE/dt is being evaluated, whereas in the radiation field there is retardation so Eq. (4) involved time-retarded quadrupole moments. Explain why there is no retardation in Eq. (10).

4. **GW Energy Loss and Orbit Shrinkage for a Binary Star System in a Circular Orbit**

a. For a Newtonian, circular binary with total mass $M = m_1 + m_2$, reduced mass $\mu = m_1 m_2 / M$, stellar separation a, and orbital angular velocity $\Omega = \sqrt{M/a^3}$ (by Kepler's laws), show that the mass quadrupole moment is

$$\mathcal{I}_{xx} = \mu a^2 \left(\cos^2 \Omega t - \frac{1}{3} \right) , \quad \mathcal{I}_{yy} = \mu a^2 \left(\sin^2 \Omega t - \frac{1}{3} \right) ,$$

$$\mathcal{I}_{xy} = \mathcal{I}_{yx} = \mu a^2 \cos \Omega t \sin \Omega t \ , \quad \mathcal{I}_{zz} = -\frac{1}{3} \mu a^2 \qquad (11)$$

b. Show that the binary's rate of energy loss is

$$\frac{dE}{dt} = -\frac{32}{5} \frac{\mu^2 M^3}{a^5} \ . \qquad (12)$$

c. Show that this energy loss causes the stars' separation to decrease with time as given by the following formula:

$$a = a_o (1 - t/\tau)^{1/4} \qquad \tau_o = \frac{5}{256} \frac{a_o^4}{\mu M^2} \qquad (13)$$

d. Show that, when rewritten in terms of the gravitational-wave frequency (which as we saw in Assignment 5 is $f = 2(\Omega/2\pi) = \Omega/\pi$, the time to merger is

$$\tau_o = \frac{(5/256)\pi^{8/3}}{f} \frac{1}{(M_{\rm chirp} f)^{5/3}} \ . \qquad (14)$$

Here

$$M_{\rm chirp} \equiv \mu^{3/5} M^{2/5} \qquad (15)$$

is the binary's *chirp mass* — which has this name because, as the binary's orbit shrinks its gravitational-wave frequency increases, i.e. "chirps", and $M_{\rm chirp}$ governs the timescale for the shrinkage and hence the timescale for the chirp. As we shall see next term, the chirp mass can be measured by LIGO or LISA with very high precision.

Some Important Applications, Largely Related to Phinney's Lecture 12 Next Week

5. **Energy and Graviton Emission Rate**

a. In the Theoretical Astrophysics Interaction Room (room 124 Bridge Annex) there is a gravitational wave generator, whose source is a small rotating dumbbell (two small masses, each weighing about 100 grams, connected by a rod with length about 20 centimeters, that spins about its center at a rate of about 30 revolutions per second. How many gravitons does this source emit per second?

b. For a binary star system consisting of two one-solar-mass stars with orbital period one hour (a type of source that Phinney will discuss in lecture 12 and that LISA will study), what is the rate of loss of energy to gravitational waves and the rate of emission of gravitons? Compare the energy loss rate to the sun's electromagnetic luminosity, $L_\odot = 3.9 \times 10^{33}$ ergs/sec.

6. **GW Frequencies Relevant to LIGO and LISA**

Consider three astrophysically important examples of binaries in circular orbits: (i) An $AmCVn$ system consisting of a one solar mass white dwarf star with a 0.01 solar mass companion, (ii) Two one-solar-mass white dwarfs or neutron stars, (iii) two 10^7 solar mass black holes.

a. For each of these, what is the gravitational-wave frequency f when the time τ_o until merger is 10^{10} years (the age of the universe)? What is the binary separation at this time and how does it compare with the sizes of the binary's objects and with the sizes of other astronomical systems with which you are familiar.

b. Answer these same questions when the time to merger is $\tau_o = 1$ year.

7. **Orbital Circularization**

Consider a binary in a highly eccentric orbit.

a. The gravitational waves are emitted almost entirely in short bursts, one each time the stars pass through their point of closest approach (their "periastron"). Explain why this is so, and estimate the frequency f of the waves and their amplitude h produced in the periastron passage — accurate to within an order of magnitude or better.

b. Estimate the GW energy emitted at each periastron passage, to within an order of magnitude.

c. Explain why this energy loss will cause the binary's apastron (maximum separation) to decrease but will not have much influence on the periastron. The result is a circularization of the orbit.

d. Estimate the time required for the binary to become nearly circular, to within an order of magnitude. Express your answer in terms of the binary's initial eccentricity e (or in terms of $1-e \ll 1$), the binary's periastron (distance of closest approach), and its masses. Give numerical values for the time to circularize for astrophysically interesting cases.