Ph 237a: Gravitational Waves 6 February 2002

WEEK 5: THE QUADRUPOLE FORMULA FOR GW GENERATION, PROPAGATION OF GW's THROUGH CURVED SPACETIME AND THE GW STRESS-ENERGY TENSOR

Lectures 8 and 9

Recommended Reading:
Note: All of this material is on the course web site.

1. Derivation of the quadrupole formula for gravitational-wave generation (beginning of Lecture 8): Charles W. Misner, Kip S. Thorne, and John A. Wheeler, *Gravitation* (Freeman, 1973), Sections 36.9 and 36.10. [A copy of this will be put on the class's web site.]

   Note: The treatment given in this section includes effects of self gravity in the source's interior, using the approach of Landau and Lifshitz, which Kip briefly discussed in his lecture. The analysis given in Kip's lecture corresponds to neglecting self gravity and correspondingly setting $t^{\mu\nu} = 0$ in the MTW analysis.
2. Wave propagation through curved spacetime (the remainder of Lecture 8 and most of lecture 9): Kip S. Thorne, "The Theory of Gravitational Radiation: an Introductory Review," in *Gravitational Radiation*, eds. N. Dereulle and T. Piran (North Holland, Amsterdam, 1983), pp. 1–57: Sections 1.2, 2.4.1, 2.4.2, and 2.5 and 2.6. 2.4.5.
3. The gravitational-wave stess-energy tensor (remainder of lecture 9): Kip S. Thorne, "The Theory of Gravitational Radiation: an Introductory Review," in *Gravitational Radiation*, eds. N. Dereulle and T. Piran (North Holland, Amsterdam, 1983), pp. 1–57: Section 2.4.5.

Possible Supplementary Reading:

4. Charles W. Misner, Kip S. Thorne, and John A. Wheeler, *Gravitation* (Freeman, 1973): Sections 35.7–35.15, including the exercises at the end of the chapter. This covers wave propagation through curved spacetime and the gravitational-wave stress-energy tensor. [This material is *not* on the course web site.]

Assignment, to be turned in at beginning of class on Wednesday 13 February by students registered in the course:

A. State what reading you have done, related to the course, during this past week.
B. Work those exercises, from the list below, that are useful for you (i.e. that are at the appropriate level for you [neither much too hard nor too easy] and that have a ratio of grunge to learning that is reasonable.
C. If A. and B. do not constitute enough to have taught you a reasonable amount about this week's topic, then do one or more of the following:
   i. If you already know a lot about this week's topic, just say so and stop.
   ii. Invent your own exercises and work them.

iii. Carry out further reading and state what you have done.
iv. Seek private tutoring from a knowledgable person about this week's topic.
v. Pursue some other method of learning about this week's topic, and state what you have done.

EXERCISES

Note: There are more exercises here than any single person is expected to work. Work only those exercises that are useful for you!

Exercises filling in the gaps in Kip's lectures

1. **Derivation of Quadrupole-moment formula**
   Carry out the full details of the derivation of the quadrupole-moment formula for a source with negligible self gravity — i.e. a source whose internal accelerations are produced by non-gravitational forces. In particular:
   a. In a Lorentz frame in flat spacetime, use the energy-momentum conservation law ${T^{\mu\nu}}_{,\nu} = 0$ to show that
   $$ {T^{00}}_{,00}x^j x^k = 2T^{jk} + (T^{lm}x^j x^k)_{,ml} - 2(T^{lj}x^k + T^{lk}x^j)_{,l} \ . \tag{1}$$
   b. Use this result to show that, in the slow-motion approximation, the standard retarded-integral formula for the gravitational-wave field
   $$ h_{jk}^{\rm TT} = 4\left[\frac{T_{jk}(\mathbf{x}'; t' = t - |\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|} d^3x'\right]^{\rm TT} \tag{2}$$
   reduces to
   $$ h_{jk}^{\rm TT} = \frac{2}{r}\left[\ddot{I}_{jk}(t-r)\right]^{\rm TT} = \frac{2}{r}\left[\ddot{\mathcal{I}}_{jk}(t-r)\right]^{\rm TT} \ , \tag{3}$$
   where I_{jk} is the second moment of the source's mass distribution, $\mathcal{I}_{jk}$ is the source's mass quadrupole moment, the dots denote time derivatives, and r is the distance from the source's center of mass to the observer. **NOTE: IN HIS LECTURE, KIP MISSED THE FACTOR 4 IN EQ. (2) AND THEREBY WROTE DOWN THE WRONG FORM FOR EQ. (3): HE WROTE $1/2r$ INSTEAD OF $2/r$.**
2. **Derivation of Geometric Optics Equations for GW Propagation**
   Carry out the full details of the derivation of the geometric-optics equations for gravitational-wave propagation. In particular, begin by expressing the Lorenz-gauge trace-reversed metric perturbation, in curved spacetime, in the form
   $$ \bar{h}_{\alpha\beta} = \Re(A_{\alpha\beta}e^{i\varphi}) \ , \tag{4}$$
   where $\Re$ means take the real part, φ is the waves' phase which varies on the very short lengthscale of the waves' reduced wavelength $\lambda\!\!\!^{-}$, and $A_{\alpha\beta}$ is the waves' amplitude which varies on a much longer lengthscale $\mathcal{L}$ (the smaller of the radius of curvature of the

waves' phase fronts and the radius of curvature of spacetime). Motivated by the form $\varphi = \omega(z - t)$ of the phase for plane waves propagating in the z direction of a local Lorentz frame, define the wave vector by $\vec{k} = \vec{\nabla}\varphi$ (so in the local Lorentz frame $k^0 = k^z = \omega$).

a. Using an argument in the local Lorentz frame, explain why $\vec{k}$ must change on the long lengthscale $\mathcal{L}$ and not on the short lengthscale $\lambda\!\!\!\!-$.

b. Show that at leading order in the small parameter $\lambda\!\!\!\!-/\mathcal{L} \ll 1$, the Lorenz gauge condition $\overline{h}^{\alpha\beta}{}_{|\beta} = 0$ reduces to the transversality condition

$$\bar{h}_{\alpha\beta}k^{\beta} = 0 \ . \tag{5}$$

c. Show that at the leading order in $\lambda\!\!\!\!-/\mathcal{L}$, the gravitational wave equation $\overline{h}_{\alpha\beta|\mu}{}^{\mu} = 0$ reduces to the statement that the wave vector is null $\vec{k} \cdot \vec{k} = 0$, and that the gradient of $\vec{k} \cdot \vec{k} = 0$ implies that $\vec{k}$ *is the tangent to a null geodesic* (the waves' *ray*).

d. Show that at the next order in $\lambda\!\!\!\!-/\mathcal{L}$, the gravitational wave equation reduces to the following transport law for the trace-reversed metric perturbation:

$$\bar{h}_{\alpha\beta|\mu}k^{\mu} = -\frac{1}{2}k^{\mu}{}_{|\mu}\bar{h}_{\alpha\beta} \ . \tag{6}$$

Note that in his lecture, Kip wrote this equation in terms of $A_{\alpha\beta}$. Explain explicitly why it can be written equally well in terms of $\bar{h}_{\alpha\beta}$ and $A_{\alpha\beta}$.

3. **Propagation Laws for h_+, $h_\times$, and their polarization tensors**
Express the trace-reversed metric perturbation in the form

$$\bar{h}_{\alpha\beta} = h_+ e^+_{\alpha\beta} + h_\times e^\times_{\alpha\beta} \ , \tag{7}$$

where $\mathbf{e}^+$ and $\mathbf{e}^\times$ are polarization tensors that are defined to be parallel propagated along the rays, $\nabla_{\vec{k}}\mathbf{e}^J = 0$ (for $J = +, \times$), and that in a local Lorentz frame of the source, near the source, have the usual components: $e^+_{xx} = -e^+_{yy} = 1$, $e^\times_{xy} = e^\times_{yx} = 1$, all other components vanish. (Here, on any chosen ray, we have oriented the coordinates so the ray points spatially in the z direction.) We do not yet know that the h_+ and $h_\times$ in Eq. (7) are the usual gravitational-wave fields measured by observers; we shall show that this is so below.

a. Show that in the local Lorentz frame of the source, our Lorenz-gauge, trace-reversed metric perturbation $\bar{h}_{\alpha\beta}$ is trace free, and therefore is equal to the metric perturbation itself, $\bar{h}_{\alpha\beta} = h_{\alpha\beta}$.

b. Use the curved-spacetime wave equation for $\bar{h}_{\alpha\beta}$ to show that it remains trace-free as it propagates, so everywhere $\bar{h}_{\alpha\beta} = h_{\alpha\beta}$. We did not have to choose our gauge so this is true, but it was convenient to do so.

c. Show, from the parallel-transport law for the polarization tensors, that $e^J_{\alpha\beta}$ always remains trace free and always satisfies $e^J_{\alpha\beta}e^{J\alpha\beta} = 2$.

d. Consider an observer far from the source, whom the waves pass. Introduce the observer's local Lorentz frame and orient its axes so the waves are propagating in the z direction, and the $+$ and $\times$ polarization axes are oriented in the usual way. Show that, by virtue of the transversality relation (5) and the relation $e^J_{\alpha\beta}e^{J\alpha\beta} = 2$, the observer's TT projection of the polarization tensors will have the usual form

$$(e^+_{xx})^{\rm TT} = -(e^+_{xx})^{\rm TT} = 1 \ , \quad (e^\times_{xy})^{\rm TT} = -(e^\times_{yx})^{\rm TT} = 1 \ , \tag{8}$$

all other components vanish. Explain why this, together with Eq. (7), implies that, as seen in the local Lorentz frame of any observer, the h_+ and $h_\times$ of Eq. (7) are the usual gravitational wave fields.

e. By inserting Eq. (7) into the propagation law (6) for $\bar{h}_{\alpha\beta}$, derive the following law for propagation of the gravitational-wave fields along the waves' rays:

$$\nabla_{\vec{k}} h_+ = -\frac{1}{2}(\vec{\nabla}\cdot\vec{k})h_+ \ , \quad \nabla_{\vec{k}} h_\times = -\frac{1}{2}(\vec{\nabla}\cdot\vec{k})h_\times \ . \tag{9}$$

4. **Gravitons**

a. In Ref. 3 of the suggested reading (above) there is a written version of the derivation Kip gave in his Lecture 9, of the Isaacson stress-energy tensor for gravitational waves. The final answer for $T^{\rm GW}_{\mu\nu}$ is given in three different forms in Eq. (2.47). Explain why the first of these forms reduces to the second in trace-free Lorenz gauge (the gauge used in Exercise 3), and reduces to the third in the local Lorentz frame of any observer. Show that the third reduces to

$$T^{{\rm GW}\,\mu\nu} = \frac{1}{16\pi}\langle h_+^{|\mu} h_+^{|\nu} + h_\times^{|\mu} h_\times^{|\nu}\rangle \ . \tag{10}$$

b. Show that in the geometric optics limit, Isaacson's gravitational-wave stress-energy tensor reduces to a sum over contributions from the two polarizations, each of which has the form

$$T^{\rm GW\ J}_{\alpha\beta} = \frac{1}{16\pi}\langle h_J^2\rangle k_\mu k_\nu \ . \tag{11}$$

Here as above, $J = +$ or $\times$.

c. These waves are carried by gravitons, each of which has a 4-momentum $\vec{p} = \hbar\vec{k}$. This means that the energy density and energy flux for gravitons with polarization J can be written as

$$T^{00}_{\rm GW\ J} = N^0_J p^0 \ , \quad T^{i0}_{\rm GW\ J} = N^i_J p^0 \ , \tag{12}$$

where N^0_J is the graviton number density and N^i_J is the graviton flux. Write down, similarly, the momentum density and the momentum flux in terms of p^μ and N^ν_J.

d. Show, from Eq. (11), that the graviton number-flux 4-vector is given by

$$N^\mu_J = \frac{1}{16\pi\hbar}\langle h_J^2\rangle k^\mu \ . \tag{13}$$

e. Show that the equations of geometric optics imply that the gravitons parallel transport their 4-momenta along their world lines, $\nabla_{\vec{p}}\vec{p} = 0$. Since their 4-momenta are tangent to their world lines and are null, this means they move along null geodesics.

f. Show that the transport law for the gravitational-wave field, Eq. (9), is equivalent to the statement that gravitons are conserved, $N^{\mu}_{J\,|\mu} = 0$.

g. Show that graviton conservation and the geodesic motion of the gravitons together guarantee conservation of energy and momentum, $T^{\mu\nu}_{\mathrm{GW}J\,|\nu} = 0$.

h. Show that graviton conservation implies that h_J decreases as $1/\sqrt{\mathcal{A}}$, where $\mathcal{A}$ is the cross sectional area of a bundle of rays along which the waves are propagating. Hint: perform the calculation in a local Lorentz frame.

i. Show that graviton conservation implies that h_J decreases as $1/r$, where r is the radius of curvature of the waves' phase fronts. Hint: perform the calculation in a local Lorentz frame.

Some Applications

5. **Gravitational Waves from an Equal-Mass Binary Star System with Circular Orbit**

Consider a binary system made of two identical stars, each with mass m and radius $R \gg m$, separated by a distance a large compared to their radii.

a. Show that the binary satisfies the slow-motion assumption (internal velocity small compared to the speed of light) and has weak gravity, $|h_{\mu\nu}| \ll 1$, so the quadrupole formula should be valid (thanks to the Landau-Lifshitz-type derivation that includes self gravity). Weak gravity and slow motion also imply that Newtonian theory is quite accurate, which means that Kepler's laws should be satisfied: the orbital angular velocity is $\Omega = \sqrt{2m/a^3}$.

b. Place the binary's center at the origin of a Cartesian coordinate system with the orbit in the x-y plane and the stars on the x axis at time $t = 0$. Show that the second moment of the mass distribution has as its only nonzero components

$$I_{xx} = 2ma^2\cos^2\Omega t = ma^2(1+\cos 2\Omega t)\;, \quad I_{yy} = 2ma^2\sin^2\Omega t = ma^2(1-\cos 2\Omega t),$$

$$I_{xy} = I_{yx} = 2ma^2\cos\Omega t\sin\Omega t = \sin 2\Omega t\;; \tag{14}$$

and thence that the second time derivative of this second moment is

$$\ddot{I}_{xx} = -\ddot{I}_{yy} = -4m(a\Omega)^2\cos 2\Omega t\;, \quad \ddot{I}_{xy} = \ddot{I}_{yx} = -4m(a\Omega)^2\sin 2\Omega t\;. \tag{15}$$

c. Introduce a spherical polar coordinate system (r, θ, ϕ) related to the Cartesian coordinates in the usual way, and denote by $\mathbf{e}_{\hat\theta}$ and $\mathbf{e}_{\hat\phi}$ the unit vectors pointing along the ϕ and θ directions. For an observer at location (r, θ, ϕ), use these basis vectors as the polarization axes, so that

$$h_+ = \frac{2}{r}\ddot{I}^{\mathrm{TT}}_{\hat\theta\hat\theta}(t-r) = -\frac{2}{r}\ddot{I}^{\mathrm{TT}}_{\hat\phi\hat\phi}(t-r)\;, \quad h_\times = \frac{2}{r}\ddot{I}^{\mathrm{TT}}_{\hat\theta\hat\phi}(t-r) = -\frac{2}{r}\ddot{I}^{\mathrm{TT}}_{\hat\phi\hat\theta}(t-r)\;. \tag{16}$$

By computing from (15) the $\hat\theta$ and $\hat\phi$ components of $\ddot I_{jk}$ and then removing the trace, obtain the TT components of $\ddot I_{jk}$, and thereby conclude that the gravitational-wave fields have the following forms. *These forms are written in a way that turns out to remain valid for a circular binary with unequal masses.*

$$h_+ = 2(1+\cos^2\theta)\frac{\mu}{r}(\pi M f)^{2/3}\cos(2\pi f t) \quad h_\times = 4\cos\theta\frac{\mu}{r}(\pi M f)^{2/3}\sin(2\pi f t)\ . \tag{17}$$

Here $f = 2(\Omega/2\pi) = \Omega/\pi$ is the waves' frequency, $\mu = m/2$ is the binary's reduced mass, $M = 2m$ is the binary's total mass, and $(\pi M f)^{2/3} = (a\Omega)^2$.

d. Show that these waveforms agree with the result, derived by dimensional analysis in Kip's introductory lectures, that the gravitational-wave amplitude has a magnitude equal to $1/c^2$ times the Newtonian gravitational potential produced by the mass equivalent of the source's internal kinetic energy.

6. **Theorem: Conservation Laws Associated with Symmetries of the Metric**

a. Consider a particle that moves along a geodesc though curved spacetime. Parametrize the geodesic by a parameter ζ defined such that $d/d\zeta = \vec p$, where $\vec p$ is the particle's 4-momentum. Show that if the particle has finite rest mass m, then ζ is related to its proper time by $\zeta = \tau/m$. If the particle is a photon or graviton and so has vanishing rest mass, m vanishes. Show that there also is no proper time lapse along the particle's world line, so τ is undefined. For such a particle ζ is a valid parameter along its world line but τ is not. Show that the geodesic equation for such a particle takes the form

$$\frac{d^2x^\mu}{d\zeta^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\zeta}\frac{dx^\beta}{d\zeta} = 0. \tag{18}$$

b. Suppose spacetime has a metric which, in some carefully chosen coordinate system, is independent of the time coordinate, so $g_{\alpha\beta,0} = 0$. Show from the geodesic equation that the component $p_0 = g_{0\mu}p^\mu$ of the particle's 4-momentum is conserved. [Similarly, if $g_{\alpha\beta,j} = 0$ for some specific $j = 1, 2, 3$, then p_j is conserved.]

7. **Gravitational Redshift of Gravitational Waves**

Consider gravitational waves traveling through the spacetime of a nonspinning black hole. In appropriate coordinates (t, r, θ, ϕ) the spacetime metric has the Schwarzschild form

$$ds^2 = -(1-2M/r)dt^2 + \frac{dr^2}{1-2M/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\ . \tag{19}$$

Here M is the hole's mass and the radial location $r = 2M$ is the hole's horizon. Far from the hole, $r \gg 2M$, the metric becomes that of flat spacetime in spherical polar coordinates.

a. Consider a family of observers who are at rest with respect to the black hole so their 4-velocities $\vec u$ all point along the time direction. Show that

$$\vec u = \vec e_{\hat 0} = \frac{1}{\sqrt{1-2M/r}}\frac{\partial}{\partial t} \tag{20}$$

b. Let the gravitational waves have a reduced wavelength small compared to the hole's size, $\lambdabar \ll 2M$, and small compared to the radii of curvature of their phase fronts. Then geometric optics is valid. Consider a graviton moving along a ray of the waves. The at-rest observers measure the graviton's energy as it passes. Explain why the energy they measure is $E = -\vec{p} \cdot \vec{u} = -\vec{p} \cdot \vec{e}_{\hat{0}} = -p_{\hat{0}}$.

c. Show that the measured energy is

$$E = \frac{-p_0}{\sqrt{1 - 2M/r}} \ . \tag{21}$$

Show that p_0 is conserved by virtue of Exercise 6. This means that as the gravitons travel to larger and larger radii r, the graviton energy measured by the at-rest observers grows smaller and smaller, i.e. it gets gravitationally redshifted by the black hole's spacetime curvature.

c. Show that, if the waves are traveling precisely radially through the black-hole spacetime, then the amplitudes of their wave fields will decrease as $1/r$, where r is the radial coordinate. Hint: consider the cross sectional area of a bundle of rays.

d. Assume that these radially traveling waves are monochromatic. Show that their phase must have the form $\varphi = \sigma(r_* - t)$, where $r_* = r + 2M \ln(r/2M - 1)$. Hint: show that the gradient of this phase function is null and has $k_0 = p_0/\hbar$ constant. Explain why this proves the desired result.

e. What is the energy E of a graviton for these waves, measured by an at-rest observer, in terms of the constant σ? What is the frequency that the observer measures?

e. Combining the results of (c) and (d), show that the radially traveling, monochromatic waves have the form

$$h_J = \frac{A_J \cos[\sigma(r_* - t) + \delta_J]}{r} \ , \tag{22}$$

where δ_J is some arbitrary constant phase factor and A_J is a constant amplitude.