Ph 237a: Gravitational Waves 30 January 2002

WEEK 4: WEAK GRAVITATIONAL WAVES IN OTHERWISE FLAT SPACETIME

Recommended Reading:

1. Blandford and Thorne, *Applications of Classical Physics*, [available on the web at http://www.pma.caltech.edu/Courses/ph136/ph136.html]: the following sections of Chapter 24 (version 0024.2) and Chapter 26 (version 0026.2).
   a. Section 24.9, "Weak Gravitational Fields"; especially Sec. 24.9.2 on "Linearized Theory".
   b. Sections 26.3.1, 26.3.2, 26.3.3, 26.3.7 on gravitational waves. Note: These sections are written from a more sophisticated viewpoint than we have taken as yet: they are treating gravitational waves that propagate through curved spacetime, not flat. However, they make extensive use of local Lorentz frames of the curved background spacetime through which the waves are propagating, and if one replaces these local Lorentz frames by global Lorentz frames of a flat background spacetime, one obtains much of the material that Kip covered in class. This replacement entails replacing every subscript "—" (gradient or covariant derivative with respect to the flat background) by a subscript comma (gradient in our background global Lorentz frame, i.e. partial derivative).

Possible Supplementary Reading:

3. Bernard F. Schutz, *A First Course in General Relativity* (Cambridge University Press, 1985 & 1990), Sections 8.3, 9.1 and 9.2
4. Charles W. Misner, Kip S. Thorne, and John A. Wheeler, *Gravitation* (Freeman, 1973), Chapter 18 on the linearized approximation to general relativity, and Sections 35.1 to 35.6 on gravitational waves.

Assignment, to be turned in at beginning of class on Wednesday 6 February by students registered in the course:

A. State what reading you have done, related to the course, during this past week.
B. Work those exercises, from the list below, that are useful for you (i.e. that are at the appropriate level for you [neither much too hard nor too easy] and that have a ratio of grunge to learning that is reasonable.
C. If A. and B. do not constitute enough to have taught you a reasonable amount about this week's topic, then do one or more of the following:
   i. If you already know a lot about this week's topic, just say so and stop.
   ii. Invent your own exercises and work them.
   iii. Carry out further reading and state what you have done.
   iv. Seek private tutoring from a knowledgable person about this week's topic.
   v. Pursue some other method of learning about this week's topic, and state what you have done.

EXERCISES

Note: There are more exercises here than any single person is expected to work. Work only those exercises that are useful for you!

Exercises filling in the gaps in Kip's lectures

1. **Electromagnetic Analogs of $h^{\rm TT}_{jk}$, h_+ and $h_\times$**
   The gravitational-wave analysis given in Kip's lectures and in the exercises that follow is closely analogous to the following treatment of electromagnetic waves.

   Consider a plane electromagnetic wave propagating in the z direction through a Lorentz frame of flat spacetime. The wave has an antisymmetric electromagnetic field tensor $F_{\mu\nu}(t-z)$ whose components are related to those of the electric and magnetic field by $F_{j0} = -F_{0j} = E_j$ and $(F_{23}, F_{31}, F_{12}) = (B_1, B_2, B_3)$.

   a. Use Maxwell's equations to verify that **E** and **B** are transverse (have vanishing z components), and that all components of $F_{\mu\nu}$ can be expressed in terms of F_{j0}; in other words, the magnetic field can be expressed in terms of the electric field. This is analogous to the transversality of the tidal forces in a gravitational wave, and to the fact that all components of the Riemann tensor for a gravitational wave are expressable in terms of R_{j0k0}.

   b. Define $A^{\rm T}_j$ by $E_j \equiv -A^{\rm T}_{j,t}$. Here and throughout we use the notation that subscripts $0, 1, 2, 3$ are equivalent to subscripts t, x, y, z. This $A^{\rm T}_j$ is the analog of $h^{\rm TT}_{jk}$ for a gravitational wave. Since the electromagnetic wave is transverse, the only nonzero components of $A^{\rm T}_j$ are $A^{\rm T}_x$ (the analog of h_+) and $A^{\rm T}_y$ (the analog of $h_\times$).

   c. Now introduce the 4-vector potential A_μ (not to be confused with $A^{\rm T}_j$), from which the electromagnetic field tensor can be constructed via $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$. Show that in Lorenz gauge, where ${A_\mu}^{,\mu} = 0$, Maxwell's equations reduce to the wave equation for A_μ, and thence (since we are considering a plane wave propagating in the z direction), A_μ is a function only of $t-z$. This is the analog of the trace reversed metric peturbation for a plane gravitational wave being a function of $t-z$ in gravitational Lorenz gauge (discussed by Kip in his lectures).

   d. Find a specific gauge-change generator $\Psi(t-z)$ that brings A_μ into a special Lorenz gauge in which $A_0 = A_z = 0$ so the that A_μ is *transverse*. Show that in this special Lorenz gauge, the spatial components of the vector potential are $A_j = {A_j}^{\rm T}$. We call this Transverse gauge or T gauge. It is the electromagnetic analog of TT gauge for a gravitational wave.

   e. Show that the T-gauge fields ${A_x}^{\rm T}$ and ${A_y}^{\rm T}$ can be obtained from the vector potential in any gauge where $A_\mu = A_\mu(t-z)$ by simple projection — i.e., by throwing way the temporal and longitudinal components of A_μ and setting $A^{\rm T}_x = A_x$ and $A^{\rm T}_y = A_y$. This is the analog of computing the components of $h^{\rm TT}_{jk}$ by projection, in any gauge where the metric perturbation is a function of $t-z$.

   f. The fields ${A_x}^{\rm T}$ and ${A_y}^{\rm T}$ depend on one's choice of reference frame. Show that when one rotates the frame's basis vectors in the transverse plane in the manner of Eq. (26.51), ${A_x}^{\rm T}$ and ${A_y}^{\rm T}$ change by

$$({A_x}^{\rm T} + i{A_y}^{\rm T})_{\rm new} = ({A_x}^{\rm T} + i{A_y}^{\rm T})_{\rm old} e^{i\psi} \ .$$

g. Show that, when one performs a boost along the z axis to a new reference frame moving at speed β with respect to the old one, the fields $A_x^{\rm T}$ and $A_y^{\rm T}$ (which are defined in terms of the electric fields measured in the two frames), are unchanged at a fixed location in spacetime; i.e. they behave like scalars. This is not true of the electric field itself! [Hint: use the result of part e.]

2. **For a Weak, Plane Gravitational Wave, All Components of Riemann are Determined by R_{j0k0}, Which is TT**

Consider a solution of the gravitational wave equation for plane waves propagating in the z direction through a global Lorentz frame,

$$R_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta}(t - z) \ . \tag{6}$$

a. By integrating the Bianchi identity with respect to time for such a gravitational wave, derive Eqs. (26.39) of Blandford and Thorne.

b. Use these relations and Riemann's symmetries to show explicitly that all components of the Riemann tensor for these waves can be expressed in terms of R_{j0k0}. This is the analog of Exercise 1.a for the electromagnetic field.

c. Then use the vacuum Einstein equations to show explicitly that R_{j0k0} is spatially transverse and trace-free (TT), i.e. it satisfies Eqs. (26.40) of Blandford and Thorne, and $R_{x0x0} + R_{y0y0} = 0$; cf. Eqs. (26.41); cf. the transversality of the electric field, Exercise 1.a.

3. **Gravitational Gauge Changes; Transformation to Lorenz Gauge**

Exercise 24.13 of Blandford and Thorne. This is analogous to Exercise 1.c for the electromagnetic field.

4. **Transformation to TT Gauge**

Consider a plane gravitational wave propagating in the z direction and analyzed in any gauge in which $h_{\alpha\beta}$ is a function only of $t - z$ (e.g. in any Lorenz gauge).

a. Exhibit a gauge transformation that brings this wave into TT gauge, so $h_{\alpha\beta}^{\rm new} = h_{\alpha\beta}^{\rm TT}$. This is analogous to Exercise 1.d.

b. Show that this gauge transformation can be achieved by TT projection — i.e., by simply throwing away the time-time and time-space and longitudinal components of $h_{alpha\beta}(t - z)$, keeping only the spatial and transverse components (those in the x-y plane, and removing the trace of these components, so

$$h_+ \equiv h_{xx}^{\rm TT} = h_{xx} - \frac{1}{2}(h_{xx} + h_{yy}) = \frac{1}{2}(h_{xx} - h_{yy}) \ ,$$

$$-h_+ = h_{yy}^{\rm TT} = h_{yy} - \frac{1}{2}(h_{xx} + h_{yy}) = -\frac{1}{2}(h_{xx} - h_{yy}) \ ,$$

$$h_\times \equiv h_{xy}^{\rm TT} = h_{yx}^{\rm TT} = h_{xy} = h_{yx} \ .$$

This is analogous to Exercise 1.e.

5. **Behavior of h_+ and $h_\times$ Under Rotations and Boosts**
   a. For a weak plane wave propagating in the z direction of a background Lorentz frame, show that h_+ and $h_\times$ transform under rotations through an angle ψ about the propagation direction in the manner of Eq. (26.51) of Blandford and Thorne. Rewrite this transformation law in terms of $\cos 2\psi$ and $\sin 2\psi$, and thereby recover the formulas that Kip gave in his Monday lecture. This is analogous to Exercise 1.f.
   b. For this same weak, plane wave, show that at any event in spacetime h_+ and $h_\times$ are invariant under boosts along the waves' propagation direction (z direction). [This is analogous to Exercise 1.g.] One way to show this is: (i) apply a Lorentz transformation to the components of the Riemann tensor, and then (ii) in each of the two reference frames construct $h_{jk}^{\rm TT}$. This is a hard way with pitfalls. A much simpler way is to use a result from Exercise 4.b that, in any gauge where the metric perturbation has the speed-of-light-propagation form $h_{\alpha\beta}(t-x)$, one can compute the gravitational wave field ${h_{jk}}^{\rm TT}$ by projection. The idea, then, is to begin in TT gauge of one of the two frames, with $h_{\alpha\beta}(t-z)$ equal to that frame's TT field, perform a Lorentz transformation of that field to take its components to the other reference frame, then use projection to extract the second frame's TT field.
6. **Motion of a Free Particle in TT Gauge**
   Consider a gravitational wave as described in TT gauge, so the spacetime metric has the form $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}^{\rm TT}(t-z)$. Consider a free particle that is at rest in this coordinate system before the wave arrives. Use the geodesic equation to show that the particle remains always at rest in this coordinate system, even while the wave is interacting with it.