WEEK 3: INTRODUCTION TO GENERAL RELATIVITY & GRAVITATIONAL WAVES

Recommended Reading:

1. Blandford and Thorne, *Applications of Classical Physics*, [available on the web at http://www.pma.caltech.edu/Courses/ph136/ph136.html]: the following sections of Chapter 23 (version 0023.2) and Chapter 24 (version 0024.2).
   a. Section 23.4, "The Stress-Energy Tensor Revisited"; also item 2 in Possible Supplementary Reading, below, if you are not already familiar with the stress-energy tensor.
   b. Sections 24.1 — 24.8. Pay special attention to Section 24.4 (in which, if you wish, you can regard the particle as having a finite rest mass m and a 4-momentum $\vec{p} \equiv m\vec{u}$ where $\vec{u}$ is its 4-velocity, so the geodesic equation $\nabla_{\vec{p}}\vec{p} = 0$ is equivalent to the one Kip gave in his Wednesday lecture, $\nabla_{\vec{u}}\vec{u} = 0$, and so $\zeta = \tau/m$). Also pay special attention to Sections 24.5, 24.6 and 24.8.

Possible Supplementary Reading:

2. Blandford and Thorne, Chapter 1, "Physics in Flat Spacetime: Geometric Viewpoint": those portions cross referenced in the recommended reading, most especially
   a. The discussion of the Levi-Civita tensor in Section 1.9
   b. Section 1.11 "Volumes, Integration, and the Gauss and Stokes Theorems",
   c. Section 1.12 "The Stress-Energy Tensor and Conservation of 4-Momentum".
3. Bernard F. Schutz, *A First Course in General Relativity* (Cambridge University Press, 1985 & 1990), Sections 6.4–6.7 and Chapter 8.
4. Charles W. Misner, Kip S. Thorne, and John A. Wheeler, *Gravitation* (Freeman, 1973), Section 8.7, Chapter 11 and Chapter 17.

Assignment, to be turned in at beginning of class on Wednesday 30 January by students registered in the course:

A. State what reading you have done, related to the course, during this past week.
B. Work those exercises, from the list below, that are useful for you (i.e. that are at the appropriate level for you [neither much too hard nor too easy] and that have a ratio of grunge to learning that is reasonable.
C. If A. and B. do not constitute enough to have taught you a reasonable amount about this week's topic, then do one or more of the following:
   i. If you already know a lot about this week's topic, just say so and stop.
   ii. Invent your own exercises and work them.
   iii. Carry out further reading and state what you have done.
   iv. Seek private tutoring from a knowledgable person about this week's topic.
   v. Pursue some other method of learning about this week's topic, and state what you have done.

EXERCISES

Note: There are more exercises here than any single person is expected to work. **You may learn useful things by reading all the exercises**, but work only those that are useful for you.

1. **Transformation of Coordinates; Fixing Four Components of the Metric**

   a. Suppose that one knows the components of a tensor $T_{\alpha\beta}$ and $T^{\mu\nu}$ in the coordinate basis associated with a coordinate system $x^\alpha(\mathcal{P})$. Introduce some other "primed" coordinate system $x^{\sigma'}(\mathcal{P})$. Then the primed coordinates can be written as functions of the unprimed coordinates, $x^{\sigma'}(x^0, x^1, x^2, x^3)$ or in abbreviated notation $x^{\sigma'}(x^\alpha)$; and similarly the unprimed coordinates can be written as functions of the primed ones, $x^\alpha(x^{\sigma'})$. Show that the components of the tensor $\mathbf{T}$ in the primed coordinate basis are related to those in the original, unprimed coordinate basis by

   $$T_{\sigma'\rho'} = \frac{\partial x^\alpha}{\partial x^{\sigma'}} \frac{\partial x^\beta}{\partial x^{\rho'}} T_{\alpha\beta} \; , \quad T_{\alpha\beta} = \frac{\partial x^{\sigma'}}{\partial x^\alpha} \frac{\partial x^{\rho'}}{\partial x^\beta} T_{\sigma'\rho'} \; .$$

   Note that the indices in these equations line up in the usual automatic way.

   b. Suppose that one knows the components $g_{\alpha\beta}$ of the metric in the coordinate basis associated with a coordinate system $x^\mu(\mathcal{P})$, and one wants to find a coordinate transformation $x^{\sigma'}(x^\alpha)$ [and its inverse $x^\mu(x^{\sigma'})$] to a new coordinate system $x^{\sigma'}(\mathcal{P})$, in which four of the metric components have the following special values: $g_{0'0'} = -1$, $g_{0'j'} = 0$ with $j' = 1', 2', 3'$. Exhibit a set of four differential equations for the four functions $x^\mu(x^{\sigma'})$ which, if satisfied, will guarantee that these special values are achieved. It is possible, quite generally, to find solutions to these four equations for the four unknowns $x^\mu(x^{\sigma'})$. It is this possibility of fixing four components of the metric however one wishes (e.g. so $g_{0'0'} = -1$, $g_{0'j'} = 0$) that forced Einstein to abandon his original guess $R_{\mu\nu} = 4\pi G T_{\mu\nu}$ for the field equation: that guess gave 10 differential equations for the 10 components of the metric, leaving no freedom to adjust 4 components at will. Note: A coordinate system in which $g_{0'0'} = -1$ and $g_{0'j'} = 0$ is called a *synchronous coordinate system.*

2. **Vacuum Einstein Equations**

   Show that in vacuum the Einstein field equations $G_{\alpha\beta} = 0$ (where $\mathbf{G}$ is the Einstein tensor) are equivalent to $R_{\alpha\beta} = 0$ (where $\mathbf{R}$ is the Ricci tensor, i.e. the contraction of the Riemann tensor on its first and third slots); i.e. show that the vacuum Einstein equations reduce to

   $$R_{\alpha\beta} \equiv R^\mu{}_{\alpha\mu\beta} = 0 \; . \qquad (1)$$

3. **Vacuum Wave Equation for Riemann Tensor when Curvature is Weak**

   Consider the Riemann curvature tensor $R_{\alpha\beta\gamma\delta}$ describing a very weak warpage of spacetime in vacuum — e.g. a gravitational wave propagating through intergalactic space. When the curvature is ignored (zero-order approximation), we can introduce a global Lorentz frame (global Minkowski coordinates) in which the metric coefficients are $g_{\alpha\beta} = \eta_{\alpha\beta}$ and the connection coefficients vanish. At first order in the curvature, we can treat $R_{\alpha\beta\gamma\delta}$ as a linear field living in this global Lorentz frame. It then has the following properties discussed in the reading and mentioned briefly in Kip's lecture: (i) It satisfies the Bianchi identity

$$R_{\alpha\beta\gamma\delta,\epsilon} + R_{\alpha\beta\delta\epsilon,\gamma} + R_{\alpha\beta\epsilon\gamma,\delta} = 0\ ; \qquad (2)$$

here the derivatives would normally be spacetime gradients ("covariant derivatives", replace comma by semicolon), but in our global Lorentz frame they reduce to partial derivatives with respect to the coordinates and thus are written with commas rather than semicolons. (ii) It satisfies the vacuum Einstein equations (1), in which the μ index is raised using the flat-space metric $\eta^{\mu\nu}$ since our reference frame is globally Lorentz. (iii) It satisfies the symmetries

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}\ , \quad R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma}\ , \quad R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}\ . \qquad (3)$$

a. By contracting the Bianchi identity on its first and fifth slots and combining with the vacuum Einstein equation, show that the Riemann tensor is divergence-free on its first slot:

$$R^{\mu}{}_{\beta\gamma\delta,\mu} = 0\ . \qquad (4)$$

b. By invoking Riemann's symmetries, show that it is divergence-free on all four slots.

c. By taking the divergence of the Bianchi identity on its last slot and using the fact that Riemann is divergence-free, derive the wave equation

$$R_{\alpha\beta\gamma\delta,\mu\nu}\eta^{\mu\nu} = \left[-\left(\frac{\partial}{\partial t}\right)^2 + \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2 \right] R_{\alpha\beta\gamma\delta} = 0\ . \qquad (5)$$

Kip will use this wave equation as a starting point for his analysis of gravitational waves next week.

4. **Vacuum Wave Equation for Riemann Tensor when Curvature is Strong** Suppose that the spacetime curvature is strong, so one cannot introduce, as a zero-order approximation, a global Lorentz frame. Then repeat the analysis of the previous exercise to obtain a wave equation of the form

$$R_{\alpha\beta\gamma\delta;\mu\nu}g^{\mu\nu} = \text{(terms involving products of the Riemann tensor with itself).} \qquad (6)$$

[Note: in order to derive this equation you will have to generalize the defining equation $A^{\alpha}{}_{;\beta\gamma} - A^{\alpha}{}_{;\gamma\beta} = -R^{\alpha}{}_{;\mu\beta\gamma}A^{\mu}$ for the Riemann tensor, to get a formula for what happens when you interchange gradient slots on a tensor of higher rank than one, e.g. a formula for $B_{\mu\nu\lambda\rho;\beta\gamma} - B_{\mu\nu\lambda\rho;\gamma\beta}$ for a fourth rank tensor **B**.] Kip will use the nonlinear vacuum wave equation (6) as a starting point, later in this class, for analyzing the propagation of gravitational waves through a strongly curved "background" spacetime.

5. **Local Lorentz Frame in Friedman Universe**
   a. Exercise 24.2 of Blandford and Thorne. Note: This exercise illustrates the fact that, in a local Lorentz frame, the metric components have their flat spacetime form in the vicinity of the spatial origin for all time, aside from corrections that are second order in the spatial distance from the origin [Eq. (24.15) of Blandford and Thorne].

6. **Second Time Derivative in a Local Lorentz Frame**
   In his lecture on Wednesday, Kip asserted that, when one evaluates the equation of geodesic deviation in the local Lorentz frame of one of the two freely falling particles, the left-hand side of the equation, $\nabla_{\vec{u}}\nabla_{\vec{u}}\vec{\xi}$ (where $\vec{u}$ is the 4-velocity of the particle whose local Lorentz frame one is using and $\vec{\xi}$ is the separation vector between particles) reduces to $(\partial/\partial t)^2\xi^\alpha$. Show that this is true using the result of the previous exercise.

7. **Stress-Energy Tensor for a Perfect Fluid**
   Exercise 23.7(a) of Blandford and Thorne.

8. **Orders of Magnitude of the Radius of Curvature of Spacetime**
   Exercise 24.7 of Blandford and Thorne.