WEEK 2: THE MATHEMATICS UNDERLYING GENERAL RELATIVITY

Recommended Reading:

1. Roger D. Blandford and Kip S. Thorne, *Applications of Classical Physics* [cited henceforth as "Blandford and Thorne"], available on the web at http://www.pma.caltech.edu/Courses/ph136/ph136.html
   a. Sections 23.1–23.3 of Chapter 23, "From Special to General Relativity" (version 0023.2)
   b. Equation (24.39) and Section 24.6 of Chapter 24, "The Field Equations of General relativity" (version 0024.2).

   Note: this introduction to the mathematics underlying general relativity assumes some prior familiarity with special relativity from a particular point of view, which is presented in Chapter 1 of Blandford and Thorne, "Physics in Flat Spacetime: Geometric Viewpoint" (available at above url). Some readers may wish to consult that chapter as they study Chapters 23 and 24; there are extensive cross references to it.

Possible Supplementary Reading:

2. Blandford and Thorne, Chapter 1, "Physics in Flat Spacetime: Geometric Viewpoint". See note on item 1 above.
3. Bernard F. Schutz, *A First Course in General Relativity* (Cambridge University Press, 1985 & 1990), Chapters 2, 3, 5, 6. Not available on the web; a few copies are on sale at the bookstore, and one copy is on reserve in Millikan Library.
4. Charles W. Misner, Kip S. Thorne, and John A. Wheeler, *Gravitation* (Freeman, 1973), Chapters 2, 3, 5, 8. Not available on the web; a few copies are on sale at the bookstore, and one copy is on reserve in Millikan Library. Warning: This textbook is terribly out of date, as are *all* other advanced textbooks on general relativity. However, that does not affect the introduction to general relativity; only the applications are out of date (black holes, gravitational waves, cosmology, experimental tests, ...).

Assignment, to be turned in at beginning of class on Wednesday 23 January by students registered in the course:

A. State what reading you have done, related to the course, during this past week.
B. Work those exercises, from the list below, that are useful for you (i.e. that are at the appropriate level for you [neither much too hard nor too easy] and that have a ratio of grunge to learning that is reasonable.
C. If A. and B. do not constitute enough to have taught you a reasonable amount about this week's topic, then do one or more of the following:
   i. If you already know a lot about this week's topic, just say so and stop.
   ii. Invent your own exercises and work them.
   iii. Carry out further reading and state what you have done.

iv. Seek private tutoring from a knowledgable person about this week's topic.
v. Pursue some other method of learning about this week's topic, and state what you have done.

EXERCISES

Note: There are more exercises here than any single person is expected to work. Work only those exercises that are useful for you!

Exercises filling in the gaps in Kip's Wednesday lecture

1. **Computation of components of a tensor**
   From the duality relation $\vec{e}^{\mu} \cdot \vec{e}_{\nu} = \delta^{\mu}_{\nu}$ and expansions of a tensor $\mathbf{T}(\_,\_,\_)$ in terms of basis vectors, e.g. $\mathbf{T} = T^{\alpha\beta}{}_{\mu} \vec{e}_{\alpha} \otimes \vec{e}_{\beta} \otimes \vec{e}^{\mu}$, deduce that the components of a tensor can be computed by inserting basis vectors into its slots and lining up the indices, e.g. $T^{\alpha\beta}{}_{\mu} = \mathbf{T}(\vec{e}^{\alpha}, \vec{e}^{\beta}, \vec{e}_{\mu})$.

2. **Raising and lowering of indices**
   From the properties of tensors discussed in exercise 1 and the definition of the metric in terms of the inner product, $\mathbf{g}(\vec{A}, \vec{B}) = \vec{A} \cdot \vec{B}$, show that indices on tensors can be raised and lowered using the metric components, e.g. $T^{\alpha\beta}{}_{\mu} = g^{\alpha\rho} g_{\mu\sigma} T_{\rho}{}^{\beta\sigma}$.

3. **Directional derivatives of bases**
   From the duality relation for bases and the definition $\nabla_{\vec{e}_{\alpha}} \vec{e}_{\beta} \equiv \Gamma^{\mu}{}_{\beta\alpha} \vec{e}_{\mu}$ of the connection coefficients, show that $\nabla_{\vec{e}_{\alpha}} \vec{e}^{\rho} \equiv -\Gamma^{\rho}{}_{\nu\alpha} \vec{e}^{\nu}$.

4. **Connection coefficients for the orthonormal basis associated with circular polar coordinates, and their use**
   In Euclidean 2-space (a flat sheet of paper) construct circular polar coordinates (r, ϕ).
   a. Show that the basis
   $$\mathbf{e}_{\hat{r}} \equiv \frac{\partial}{\partial r}, \quad \mathbf{e}_{\hat{\phi}} \equiv \frac{1}{r} \frac{\partial}{\partial \phi}$$
   is orthonormal; i.e. in this basis the components of the metric are the Kronecker delta.
   b. By drawing pictures, deduce the values of all the connection coefficients for this basis.
   c. Let $A^{\hat{\alpha}}$ be the components of a vector field $\mathbf{A}$ in this basis. Using your connection coefficients, derive a formula for the divergence of $\mathbf{A}$, $\boldsymbol{\nabla} \cdot \mathbf{A} = A^{\hat{\alpha}}{}_{;\hat{\alpha}}$ in terms of partial derivatives of the components of $\mathbf{A}$. Your answer should be the familiar formula
   $$\boldsymbol{\nabla} \cdot \mathbf{A} = \frac{1}{r} \frac{\partial (r A^{\hat{r}})}{\partial r} + \frac{1}{r} \frac{\partial A^{\hat{\phi}}}{\partial \phi} \ .$$

5. **Components of gradient of a tensor in terms of connection coefficients**
   By the same technique as is used in Eq. (23.29) of Blandford and Thorne, derive an expression for $F^{\alpha}{}_{\beta;\mu}$ in terms of $F^{\alpha}{}_{\beta,\mu}$ and the components of $\mathbf{F}$ and the connection coefficients of the chosen basis.

6. **Components of commutator of two vector fields**
   Let $\vec{A}(\mathcal{P})$ and $\vec{B}(\mathcal{P})$ be two vector fields. In an arbitary basis (which might or might not be a coordinate basis), derive a formula for the components of the commutator $[\vec{A}, \vec{B}]$ in terms of the components A^α and B^β, their derivatives along the basis vectors, $A^\alpha{}_{,\mu}$ and $B^\beta{}_{,\nu}$, and the basis's commutation coefficients $c_{\alpha\beta\gamma}$. In a coordinate basis your result should reduce to the one given by Kip in his lecture; cf. Eq. (23.24) of Blandford and Thorne.

7. **Formula for components of Riemann tensor in an arbitrary basis**
   Exercise 24.8 of Blandford and Thorne

8. **Formula for components of Riemann tensor in a local Lorentz frame**
   Show that in a local Lorentz frame in spacetime, the components of the Riemann tensor are given by Eq. (24.51) of Blandford and Thorne.

Additional Exercises

8. **Practice with frame-independent tensors**
   Exercise 23.3 of Blandford and Thorne.

9. **Practice with index shuffling**
   Let **F** be a second-rank tensor that is antisymmetric under interchange of its slots, i.e. $F_{\alpha\beta} = -F_{\beta\alpha}$, and that satisfies the relation

$$F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} = 0 \ .$$

   Define J^α by $F^{\alpha\beta}{}_{;\beta} = 4\pi J^\alpha$, and define

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) \ .$$

   (Actually, the electric and magnetic fields can be embodied in an antisymmetric field tensor; if **F** is that tensor, then $\vec{J}$ is the charge-current 4-vector and **T** is the electromagnetic stress-energy tensor.) Show that

$$T^{\alpha\beta}{}_{;\beta} = -F^{\alpha\beta}J_\beta \ .$$

   Actually, this equation describes the the rate at which energy and momentum are transfered between the electromagnetic field and the charge-current distribution with which it interacts.