Ph 237b: Gravitational Waves 10 April 2002

WEEK 12: OVERVIEW OF REAL INTERFEROMETERS; THERMAL NOISE

Lecture 21 by Alan Weinstein [Real Interferometers]; Lecture 22 by Phil Willems [Thermal Noise]

Recommended Reading Related to These Lectures:

1. *Overview of Real Intereferometers:* There is no good overview reference, of reasonable length, that I am aware of. As a good substitute, you are encouraged to study the pdf slides from Alan Weinstein's lecture, and simultaneously play the video of the lecture so as to hear Weinstein's words associated with each slide. [I, Kip, apologize for the quality of the video; the ceiling projector did not work and the backup portable projector, which we used, produced a very dim and low quality image. However, the image should be sufficient to indicate which slide Weinstein is talking about; the sound of his talking is clear; and the pdf slides are clear, detailed, and very nicely done.] Most of the issues in this lecture will be covered in greater detail later; this introduction will prepare you well for the later lectures.

2. *Thermal Noise:*

a. Roger D. Blandford and Kip S. Thorne, *Applications of Classical Physics*, text for Ph136, on the web at http://www.pma.caltech.edu/Courses/ph136/ph136.html ("Blandford & Thorne"): Section 5.6.1 on the Fluctuation-Dissipation Theorem; also exercises 5.7, 5.8, 5.10 at the end of Section 5.6. This contains a careful presentation and proof of the fluctuation dissipation theorem in the general Callen-Welton form (see Ref. 4.a below for the original Callan-Welton paper). The Blandford-Thorne proof of the theorem is much simpler than that given by Callen and Welton. Exercise 5.10 derives Yuri Levin's "direct method" of applying the fluctuation-dissipation theorem to a GW interferometer. [Note: Blandford and Thorne use the notation $G_x(f)$ for the spectral density of a random process $x(t)$, whereas it is conventional for gravitational-wave physicists to denote this by $S_x(f)$ or $x^2(f)$.]

b. Peter Saulson, *Fundamentals of Interferometric Gravitational Wave Detectors* (World Scientific, 1994), Chapter 7, "Thermal Noise". This chapter is available on the class website and in harc copy on a shelf on the east wall of the Theoretical Astrophysics Interaction Room (the lounge between East Bridge and West Bridge). It is a very nice and readable introduction to thermal noise but is out of date in not including Levin's direct method; see Ref. 2.a, above and Ref. 4 below. In particular, it expresses the test-mass internal thermal noise as a sum over contributions from normal modes [Section 7.9] — an approach that is much more complicated than Levin's direct method and that gives correct results only if the dissipation is homogeneously distributed inside the test mass.

Supplementary Reading

3. *Overviews of Real GW Interferometers.* The following references are long but good overviews of interferometers. They are supplementary rather than recommended because of their length:

a. Peter Saulson, *Fundamentals of Interferometric Gravitational Wave Detectors* (World Scientific, 1994). Chapter 7 is recommended; Ref. 2.b above. *Comment:* This is a beautiful, though expensive and slightly out of date book. I recommend it strongly to people who want to do gravitational-wave research.

b. Martin W. Regehr, *Signal Extraction and Control for an Interferometric Gravitational Wave Detector*, unpublished PhD thesis, Caltech, August 1, 1994. This covers many but by no means all aspects of GW interferomters, and it is clearly written. If I can get an electronic version of it, I will put it on the course website.

4. *Callen & Welton on Thermal Noise:* The following two papers developed the general formulation of the fluctuation dissipation theorem and also developed other very general and useful theorems about thermal noise. These are real classics.

a. Herbert B. Callen and Theodore A. Welton, "Irreversibility and generalized noise", *Physical Review*, **83**, 34–40 (1951); available on the web at the Physical Review site. The terminology and notation in this paper are quite different from what is now standard. Equations (4.8) in the quantum regime and (4.11) in the classical limit describe the mean square value $\langle V^2 \rangle$ of the fluctuating "generalized force" V in terms of an integral over the real part $R(\omega)$ of a complex generalized impedance $Z(\omega)$. In modern language, one switches from ω to $f = \omega/2\pi$ and thereby rewrites (4.11) as $\langle V^2 \rangle = 4kT \int R(f) df$, and one then identifies the contribution at frequency f as the "Spectral density" of V:

$$G_V(f) \equiv S_V(f) \equiv \tilde{V}^2(f) = 4kTR(f); \tag{1}$$

and similarly for the quantum-regime formula (4.8).]

b. H.B. Callen and R.F. Greene, *Physical Review*, **86**, 702 (1952); available on the web at the Physical Review site. This is a less well known paper than Callen and Welton; it generalizes the ideas in Callen and Welton.

4. *Levin's Direct Method for Evaluating Thermal Noise:* Yuri Levin, "Internal thermal noise in the LIGO test masses: A direct approach", *Physical Review D*, **57**, 659–663 (1998); available on the web at the Physical Review web site; also available at http://xxx.lanl.gov/abs/gr-qc/9707013 . Levin was a theoretical physics graduate student at Caltech when he developed this approach to thermal noise. This is one of many contributions made by theorists to gravitational wave detection.

5. *Thermoelastic Noise in Test Masses:* Y. T. Liu and K. S. Thorne, "Thermoelastic noise and homogeneous thermal noise in finite sized gravitational-wave test masses," *Physical Review D*, **62**, 122002 (2000); available on the Physical Review web site and at http://xxx.lanl.gov/abs/gr-qc/0002055 . This paper illustrates the use of Levin's direct method. Of greatest interest are

Sections I, II, and III, which lay the foundations for the computations and carry them out for interferometers whose beam spot sizes are small compared to the test-mass diameter. Thermoelastic noise was identified by Vladimir Braginsky and colleagues as a serious issue for LIGO-II: V.B. Braginsky, M.L. Gorodetsky and S.P. Vyatchanin, *Physics Letters A*, **264**, 1 (1999).

Assignment, to be turned in at beginning of class on Wednesday 17 April by students registered in the course:

A. State what reading you have done, related to the course, during this past week.

B. Work those exercises, from the list below, that are useful for you (i.e. that are at the appropriate level for you [neither much too hard nor too easy] and that have a ratio of grunge to learning that is reasonable.

C. If A. and B. do not constitute enough to have taught you a reasonable amount about this week's topic, then do one or more of the following:

- **i.** If you already know a lot about this week's topic, just say so and stop.
- **ii.** Invent your own exercises and work them.
- **iii.** Carry out further reading and state what you have done.
- **iv.** Seek private tutoring from a knowledgable person about this week's topic.
- **v.** Pursue some other method of learning about this week's topic, and state what you have done.

EXERCISES

1. Derivation of Levin's Direct Method for Interferometer Thermal Noise. Exercise 5.10 of Blandford and Thorne.

2. Various Forms of Thermal Noise and their Physical Origins

- **a.** Explain why Levin's formula (5.173) is a valid way to compute the spectral density of almost all forms of thermal noise that influence the gravitational-wave signal, not just noise associated with internal motions of the mirror. Examples are given in the next part of this exercise.
- **b.** Each form of thermal noise in the gravitational-wave signal arises from some physical process that couples fluctuations of a "thermal bath" into motions of the front of the mirror — motions measured by the laser beam. That same physical process gives rise to a portion of the dissipation $W_{\rm diss}$ when the mirror is driven by Levin's fluctuating force. Here is a list of dissipation mechanisms in Levin's thought experiment. For each of these try to figure out the physical nature of the corresponding thermal noise in the gravitational-wave data. The answer for the first dissipation mechanism is given, as an example.

(i) *Bulk thermal dissipation:* Imperfections (dislocations of atoms, impurities, ...) distributed homogeneously in the mirror substrate (the fused silica or sapphire) convert oscillating elastic energy, in the Levin thought experiment, into heat. [Answer for this first mechanism: These same imperfections produce a weak coupling between normal modes of oscillation of the test mass. Those normal modes (the heat bath), each with a time averaged energy kT, feed energy randomly back and forth to each other via that weak coupling; as a result the amplitude and phase of each normal mode changes randomly but slowly. Although each normal mode has a frequency high compared to the GW frequencies (~ 10 kHz or higher, compared to $f_{\rm GW} \sim 10$ to 1000 Hz), its slow and random changes of amplitude and phase produce tiny motions of the mirror's front face, in the GW frequency band. It is these face motions that the laser beam measures, and that thereby show up as noise in the interferometer's output.] Note: this is the "Test mass thermal noise" graphed for LIGO-I in Weinstein's slide 1.

(ii) *Thermoelastic dissipation:* The oscillating squeeze and stretch of the substrate material causes an oscillating, inhomogeneous temperature distribution; heat flows down the temperature gradient; and that heat, flowing from high temperature regions to low temperature regions, produces an increase of entropy and a corresponding conversion of oscillation energy into additional heat. [This dissipation process is analyzed quantitatively in Ref. 5.]

(iii) *Mirror coating dissipation:* Imperfections in the mirror coating material, and in the interface between the alternating layers of coating material, produce heat when the coatings are squeezed and sheared in response to Levin's oscillating force.

(iv) *Wire rubbing dissipation:* The mirrors in LIGO-I are hung by wires that are stretched around them, as shown in Weinstein's slides 28 and 31. Levin's oscillating force causes the test mass to swing and thence causes the wires to slide a bit on the test mass. Although the wires are in a carefully cut groove and are coated with a once-secret Russian substance, pork fat, the rubbing produces a bit of heat.

(v) *Suspension-wire dissipation:* As a mirror swings under the action of Levin's oscillatory driving force, the suspension wires bend, mostly near their attachment to the suspension block [see Weinstein's slide 31]. This oscillatory bending heats the wires. [Note: the thermal noise associated with this dissipation mechanism is shown as a function of frequency for LIGO-I interferometers in Weinstein's slide 1; it is the curve labeled "suspension thermal".]

(vi) *Magnetic controller dissipation:* As a mirror swings in response to Levin's oscillatory force, the tiny control magnets attached to it move in and out of the control system's coils [Weinstein's slides 32 and 33]. This motion of the magnets induces an emf in the coils, causing electric current to flow through them; that current passes through electrical resistance in the coils' circuits, producing heat.

c. Try to identify at least one other dissipation mechanism that could be analyzed by Levin's direct method.

d. There are a few forms of thermal noise that cannot be analyzed by Levin's direct method. Try to identify one or more. [Here are the names of some answers and references; these names don't reveal the physical mechanisms; identifying them is your job: (i) "photo-thermal noise", identified and named by Braginsky, Gorodetsky and Vyatchanin, Ref. 5; (ii) "Thermorefractive noise"; I don't references, nor who first pointed out this noise.]

3. **Application of Levin's Method to Suspension Thermal Noise.** Consider the *suspension thermal noise* whose associated dissipation is described in Exercise 2.b.(v). As the wire flexes, the elastic restoring force that it exerts on the test mass is given by $-k_{\rm el}x$, where x is the horizontal position of the test mass. Gravity produces a restoring force $-k_{\rm grav}x$, where $k_{\rm grav} = mg/\ell$, with m the mass, g the acceleration of gravity, and ℓ the wire's length. See Sec. 7.8 of Saulson, Ref. 2.b. The wire's material is slightly lossy, so in the frequency domain its spring constant has a small imaginary part, $ik_{\rm el}\phi_{\rm el}$. (Saulson denotes $\phi_{\rm el}$ by $\phi_{\rm wire}$.)

   a. Use Levin's direct method to derive the spectral density $S_h(f)$ of the suspension thermal noise. [Remember that there are four test masses. It is reasonable to expect their noises to be uncorrelated. Why?] Plot $\sqrt{S_h(f)}$ as a function of frequency using a reasonable choice for the parameters, corresponding to LIGO-I interferometers. Make two plots, one assuming that the material's dissipation is structural and the other viscous. For comparison, Weinstein plots the "official" expected suspension thermal noise for LIGO-I in his slide 1.

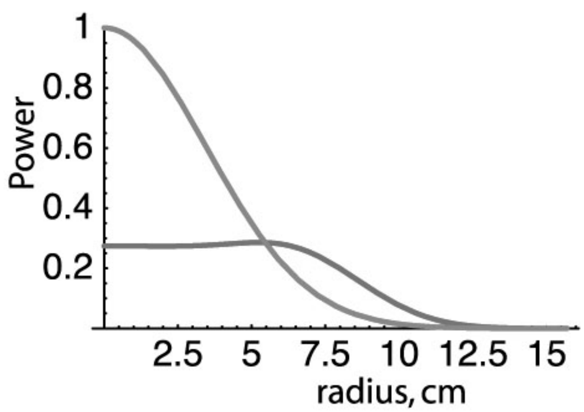
   b. Explain why the thinner the wire (for fixed mass of mirror), the smaller $k_{\rm el}$ and thence the smaller the suspension thermal noise. Thinner wires, however, will stretch more under the weight of the mirror, and if they stretch too much, they will break. Moreover, as they near their breaking strain, they begin to exhibit random sudden jerks of strain release that will make the test mass move suddenly, producing noise in the gravitational-wave signal. Braginsky, who has identified this as a serious issue, calls this and other similar noises "excess noise", since the jerks are so rare that they do not show up in the noise spectral density (but nevertheless are frequent enough to be potentially troublesome). Recent and current research in Braginsky's group (Moscow, Russia) and in Saulson's group (Syracuse, NY) is aimed at quantifying this excess noise, thereby telling us how thin the wires can be made in LIGO-I and LIGO-II.

   c. Show that the suspension thermal noise scales as the square of the wire diameter, or equivalently scales inversely with the strain of the wire. Hint: use your formula for the suspension thermal noise together with either (i) Saulson Sec. 7.8 [where $I = \int x^2 dA$ with the integral taken over the wire cross sectional area with and x the distance on the x axis away from the wire center], or (ii) the results of Exercise 10.10 of Blandford and Thorne (Chapter 10).

4. *Control of Thermoelastic Noise.* The dissipation mechanism associated with "thermoelastic noise" was described in Exercise 2.b.(ii). The corresponding fluctuational process leads to little bumps and dimples in the surface of the mirror, the most troublesome of which fluctuate in height at frequencies near the minimum of the LIGO-II noise curve, $f \sim 100$ Hz. If you did not deduce this as part of your answer to Exercise 2.b.(ii), return to that exercise and redo it, explaining how these fluctuating bumps and dimples arise.

a. The laser beam averages over these bumps and dimples. Explain why adjacent bumps and dimples should be correlated in such a way that there is a strong averaging out of their influence on the gravitational-wave signal, if the beam's intensity distribution (energy flux) is constant (spatially independent) at the bump and dimple locations. Explain why this implies that the larger the beam, the smaller the thermoelastic noise. That, indeed, is the case: As Willems discussed in his lecture, for thermoelastic noise, $S_h(f)$ scales as $1/r_o^3$, where r_o is the spot size; see Eq. (18) of Liu and Thorne, Ref. 5.

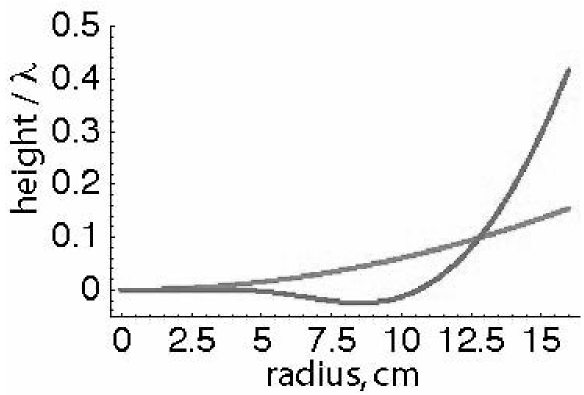
b. If sapphire is used for the substrate in LIGO-II interferometers, then thermoelastic noise and shot noise may be the dominant noise sources near the minimum of the noise curve. For this reason the thermoelastic noise will be reduced as much as possible in LIGO-II by making the beams' spot sizes as large as possible in the arm cavities. We can reduce the thermoelastic noise even further by reshaping the beams in the arm cavities, so instead of being Gaussians like the tall curve below, they have flattened (*flat-topped*) intensity distributions like the short curve below. (These flat-topped beams will average the bumps and dimples far better than a Gaussian beam does.)



Here the dimensions are appropriate for a LIGO-II sapphire mirror, which is planned to have a diameter of about 16 cm. A computation of thermoelastic noise $S_h(f)$ using Levin's direct method gives a result, for the largest acceptable flat-topped beam, that is nearly a factor 4 smaller than $S_h(f)$ for the largest acceptable Gaussian beam — a very significant improvement. See the pdf slides, on the web at

http://www.ligo.caltech.edu/docs/G/G010333-00/

from a lecture describing recent work of Erika d'Ambrosio, Richard O'Shaughnessy, and me at Caltech together with Braginsky, Sergey Strigin and Sergey Vyatchanin in Moscow. The flat-topped beam must be an eigenfunction of the arm cavities, and to achieve this we must reshape the mirrors so they are no longer segments of spheres. The curves below show the cross sections of the mirrors that are required to support the flat topped beam and the Gaussian beam depicted above:



For the flat-topped beam the mirror must have a bump at the center with thickness a few percent of a wavelength of light and radius about 5cm, and outside about 10cm radius it must have a turned-up lip of height about 1/2 wavelength. In the remainder of this exercise, you will construct a flat topped beam mathematically, then use it to compute the shapes of the mirrors needed to support it.

c. Build the flat-topped mode, inside the cavity, as a superposition of minimum-spot-sized Gaussian beams — i.e., Gaussian beams with waists at the arm's center, and with the smallest possible spot sizes at the mirrors. Since each Gaussian beam satisfies the equations of paraxial Fourier optics, so will your superposition of Gaussians. Show that each of the minimum-spot-sized Gaussian beams that you use in your construction has an electric field, at the transverse plane of each mirror, that is given by [aside from a multiplicative constant]:

$$\psi_o(\varpi) = e^{-(1+i)\varpi^2/2} \qquad (1)$$

where the radius ϖ from the beam's centerline is measured in units of $b \equiv \sqrt{\lambda L/2\pi} = 2.6$ cm, with $\lambda = 1\mu$m the wavelength of the light and $L = 4$ km the cavity arm length. Henceforth measure all transverse lengths in units of b.

d. Build the flat-topped mode mathematically by superposing minimum-spot-sized Gaussian beams with their center lines uniformly distributed inside a radius $D = 4$ (in units of b), so at the transverse plane of each mirror the electric field is

$$\psi(\varpi, D) = \int_0^D \int_0^{2\pi} \exp\left\{-(1+i)[(\varpi - \varpi' \cos\phi')^2 + (\varpi' \sin\phi')^2]/2\right\} d\phi' d\varpi' \ . \qquad (2)$$

Here the integral is over the locations (ϖ', ϕ') of the beams' centerlines. For definiteness ψ is evaluated in this formula at $\phi = 0$, but it is axisymmetric (since the Gaussian beam distribution is axisymmetric), so it would have the same value independently of the choice of ϕ. Using Mathematica or otherwise, perform the angular integral to get an expression for ψ that involves a single radial integral with a modified Bessel function of order zero in the integrand.

e. Evaluate this radial integral numerically, and plot the resulting intensity $I(\varpi, 4) = |\psi(\varpi, 4)|^2$. Your plot should look like the flat topped beam shown above.

f. From the fact that the beam's phase must be constant over the surface of a mirror, compute numerically the shape of the mirrors that are required to support this flat-topped mode.

Your plot should look like the mirror shape shown above. This type of mirror is being explored as an option for LIGO-II.