Ph 237b: Gravitational Waves 3 April 2002

WEEK 11: PHYSICS UNDERLYING INTERFEROMETRIC GW DETECTORS
Lectures 19 and 20 by Thorne

Reading Related to These Lectures:

Items in bold are recommended; others are supplementary.

1. *Proper reference frame of an accelerated observer.* C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation*:

- **a.** Chapter 6, **especially Sec. 6.6**. This treats such reference frames in flat spacetime and obtains the metric in Eq. (6.18)
- **b.** Sec. 13.6. This is a sophisticated treatment in curved spacetime, including not only acceleration of the reference frame but also rotation. The final metric, to first order in distance from the origin, is Eq. (13.71)
- **c. Secs. 37.1 and 37.2 including especially Box 37.1.** This is a careful treatment of the interaction of gravitational waves with an accelerated detector (e.g. one on the earth's surface), and concludes that the detector can be analyzed using a non-general-relativistic viewpoint (just augment the local Lorentz viewpoint with the "acceleration of gravity" and with the tidal force of the gravitational waves).

2. *Paraxial Fourier Optics.* R.D. Blandford and K.S. Thorne, *Applications of Classical Physics*, text for Ph136, on the web at http://www.pma.caltech.edu/Courses/ph136/ph136.html . Henceforth this reference will be cited as "Blandford and Thorne".

- **a. Section 7.2: the Helmholtz-Kirchoff Integral which underlies paraxial optics.**
- **b. Section 7.5.2: the propagator (called a "Point Spread Function").**

3. *Gaussian Beams.*

- **a. Blandford and Thorne, Sec. 7.5.5.**
- **b.** For a much more detailed treatment, see A.E. Siegman, *An Introduction to Lasers and Masers*, (McGraw Hill 1971), Section 8.2; on reserve in Millikan Library.

4. *Modes of Optical Cavities with Spherical Mirrors.* Sections 8.3 and 8.4 of Siegmann, Ref. 3.b.

5. *Random Processes.* The following portions of Blandford and Thorne, Chapter 5:

- **a.** Fourier transforms and spectral density: Section 5.3, **especially the segment between Eq. (5.27) and (5.35)**.
- **b.** Filtering and its influence on spectral density, and shot noise: **Section 5.5.**

6. *Optical Elements: Mirrors, Reciprocity Relations, Etalons and Fabry-Perot Cavities.* Section 8.4 of Blandford and Thorne.

7. *Description and Analysis of an Idealized Interferometer.* **Section 8.5 of Blandford and Thorne.**

Assignment, to be turned in at beginning of class on Wednesday 10 April by students registered in the course:

A. State what reading you have done, related to the course, during this past week.

B. Work those exercises, from the list below, that are useful for you (i.e. that are at the appropriate level for you [neither much too hard nor too easy] and that have a ratio of grunge to learning that is reasonable.

C. If A. and B. do not constitute enough to have taught you a reasonable amount about this week's topic, then do one or more of the following:

- **i.** If you already know a lot about this week's topic, just say so and stop.
- **ii.** Invent your own exercises and work them.
- **iii.** Carry out further reading and state what you have done.
- **iv.** Seek private tutoring from a knowledgable person about this week's topic.
- **v.** Pursue some other method of learning about this week's topic, and state what you have done.

EXERCISES

Exercises

1. Proper Reference Frame of an Accelerated Gravitational-Wave Detector. The spacetime metric in an earth-based laboratory has the following form, to high accuracy:

$$ds^2 = -(1 + 2gz)dt^2 + dx^2 + dy^2 + dz^2 \ . \tag{1}$$

a. A particle moves through this laboratory freely (along a geodesic), at speed small compared to the speed of light. Show that the particle's geodesic equation reduces to the familiar Newtonian form $d^2x/dt^2 = d^2y/dt^2 = 0$, $d^2z/dt^2 = -g$.

b. A gravitational wave with + polarization propagates vertically through this laboratory. In the laboratory's proper reference frame, the spacetime metric including the wave's influence has the following form (which can be deduced with the help of Eq. (13.73) of MTW, Ref. [1] above:

$$ds^2 = -[1 + 2gz + (y^2 - x^2)\ddot{h}_+(t - z)]dt^2 + dx^2 + dy^2 + dz^2 \ . \tag{2}$$

Here we have omitted some contributions to g_{0j} and g_{jk} that are of order $\ddot{h}_+|\mathbf{x}|^{\mathbf{2}}$ that will be explored in part c. An interferometric gravitational wave detector with arm length very small compared to the wave's wavelength is driven by the gravitational wave. Consider two test masses, one at the end of the x-arm of the interferometer (i.e., on the x axis) and the other at the end of the y arm. These test masses are suspended by a wire so they can move horizontally but not vertically. Derive the equations of motion for their horizontal motion, and show that they move in just the manner you would expect for a gravitational-wave interferometer that is falling freely.

c. Show that the contributions, of order $\ddot{h}_+|\mathbf{x}|^{\mathbf{2}}$, to g_{0j} and g_{jk} have a negligible influence on the test masses' motion.

2. Eigenquation for modes of a symmetric optical cavity. Consider an optical cavity with two identical, perflectly reflecting mirrors separated by distance L. Mirror 1 at $z = -L/2$ has a reflecting surface with shape $\delta z = \zeta(\mathbf{x}_1)$, where $\mathbf{x}_1$ is transverse vectorial position; and mirror 2 at $z = +L/2$ has the same shape but inverted, $\delta z = -\zeta(\mathbf{x}_2)$. This cavity is excited in an eigenmode with x polarization, and this mode, like the mirror configuration, is symmetric about the cavity's center plane $z = 0$. Denote by $\psi(\mathbf{x_1})$ the x component of the electric field evaluated in the transverse plane of mirror 1.

a. Show that this field satisfies the following integral equation:

$$\psi(\mathbf{x}_2)\exp[-i2k\zeta(\mathbf{x}_2)] = \int \frac{-ik}{2\pi L} e^{ikL} \exp\left[\frac{-ik(\mathbf{x}_1 - \mathbf{x}_2)^2}{2L}\right] \psi(\mathbf{x}_1) d^2x_1 \; . \qquad (3)$$

Explain this equation physically (geometrically).

b. Show that a Gaussian beam satisfies this eigenequation if its spherical phase fronts have the same radii of curvature as the two mirrors, at the mirrors' locations, and if the mirror separation L is adjusted appropriately by a fraction of a half wavelength of the light.

Comment: The eigenequation (3) describes the connection between the shapes of the mirrors and the distribution of the light over their surfaces. If we want to design a light beam that has a specific light distribution, this equation tells us how to configure the mirrors so that light which passes through the cavity will come out with very nearly the desired shape. We shall meet this mirror-design task later in the course, when we discuss the control of thermoelastic noise in LIGO.

3. Excitation, Reflection and Transmission of an Optical Cavity. Consider an optical cavity made of spherical mirrors that have reflectivities and transmissivities (r_1, t_1) for the left mirror and (r_2, t_2) for the right mirror. For simplicity assume vanishing losses, so $r_j^2 + t_j^2 = 1$. For definiteness, adopt the convention that the reflectivity for light that bounces off the exterior face of a mirror is positive, and that for light that bounces off the interior face (from inside the cavity) is negative. A Gaussian beam with wave number k impinges on the cavity from the left. Its transverse radius of curvature and spot size are adjusted so the surfaces of constant phase match the mirrors' surfaces perfectly. Henceforth ignore the transverse field distribution and focus solely on the value of the field along the optic axis. Also neglect the mode's Guoy phase, and ignore the modest changes in amplitude due to the changing spot size as the wave propagates — i.e., idealize the wave as planar.

a. Explain why, in analyzing the cavity's performance (as we shall do), it is reasonable to neglect the Guoy phase and the slowly changing amplitude.

b. Denote by ψ_1 the field (on the optic axis) of the incoming light when it hits mirror 1; denote by ${\psi'}_1$ the field reflected from mirror 1; denote by ψ_2 the field that emerges from the cavity, out of mirror 2; and denote by ψ_0 the field propagating to the right, inside the cavity. Derive general expressions for the ratios of reflected to incoming wave, ${\psi'}_1/\psi_1$, transmitted to incoming wave, ψ_2/ψ_1, and interior to incoming wave ψ_0/ψ_1.

c. Suppose that the two mirrors are identical, as for LIGO's mode cleaning cavity.

- **i.** From the results of part b, derive expressions for the ratio of transmitted to incoming light flux, $I_2/I_1 = |\psi_2/\psi_1|^2$, and reflected to incoming flux, $I_1'/I_1 = |\psi_1'/\psi_1|^2$. Show that energy is conserved: $I_2/I_1 + I_1'/I_1 = 1$.
- **ii.** Using Mathematica or some other computer tool, plot I_2/I_1 as a function of kL for various values of the mirror's power reflectivity, $(r_1)^2 = (r_2)^2 \equiv R$, running from 0 to 1. Your graphs should look something like Fig. 8.8 of Blandford and Thorne.
- **iii.** Similarly, compute and plot the ratio of light flux inside the cavity to that impinging on the cavity, as a function of kL for various R.
- **iv.** Show analytically that, if the cavity length L is held fixed, then the resonances are separated in angular frequency by $\Delta\omega_e = c\delta k = \pi c/L$. Show that for a cavity with the LIGO arm length of 4km, this separation (the cavity's free spectral range) is $\Delta\omega_e/2\pi \simeq 37$kHz. Show further that, when R is very close to unity, the resonances have half widths $\delta\omega_e$ given by $\delta\omega_e = \Delta\omega_e/F$ where $F \equiv \pi/(1-R)$ is the cavity's "finesse".

d. Suppose that the right mirror is perfectly reflecting, as for an arm cavity of an idealized LIGO interferometer. Then all the input light emerges from the input (left) mirror.

- **i.** From the results of part b, derive the phase shift ϕ put onto the reflected light by the cavity. (This phase shift is defined by $\psi_1'/\psi_1 \equiv e^{i\phi}$.)
- **ii.** Plot the phase shift ϕ as a function of kL. Your plot should show sudden changes of phase as kL passes through each cavity resonance.
- **iii.** Show analytically that in the vicinity of each resonance, small changes δL of cavity length (at fixed wave number) produce small changes of light phase $\delta\phi$ given by

$$\delta\phi = B \times 2k\delta L\ , \quad \text{where } B \equiv \frac{4}{1-R}\ , \tag{4}$$

  and $R = (r_1)^2$ is the left mirror's reflectivity. Explain why B can be interpreted as the average number of round trips the light makes in the cavity. Notice that B is approximately equal to the cavity finesse.

4. Filtering Action of a Mode Cleaning Cavity. Suppose that a mode cleaning cavity is driven by light whose frequency ω_e wanders randomly (due to laser noise) in the vicinity of a cavity resonance ω_o. Assume that the frequency range over which ω_e wanders is small compared to the cavity's free spectral range $\Delta\omega_e$ but large compared to its resonance half-width $\delta\omega_e/F$. Denote by $S_{E_1}(f)$ the spectral density of the light's electric field, $E_1 = \Re(\psi_1 e^{-i\omega_e t})$.

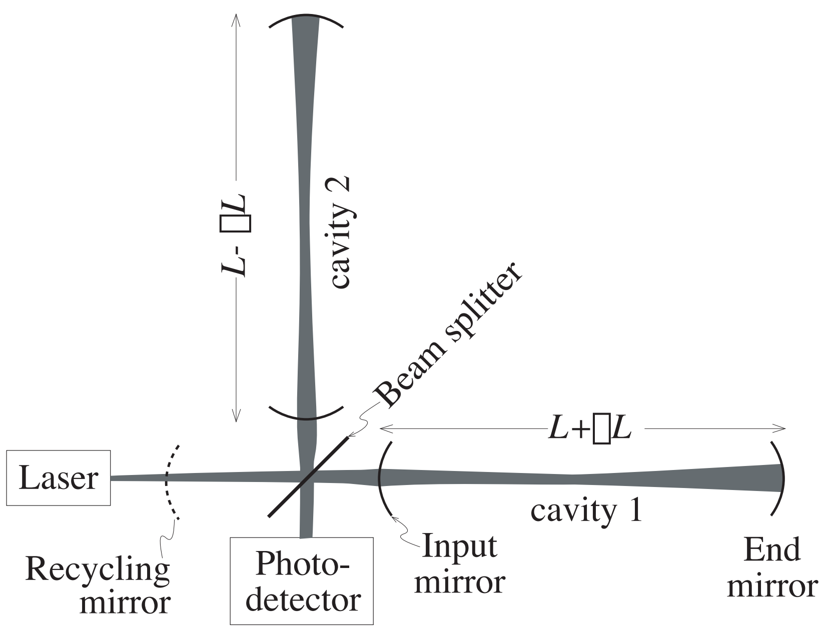
a. The cavity filters the incoming light. Show that the filtered light (emerging from the cavity's right, output mirror) has spectral density $S_{E_2}(f) = S_{E_1}(f_o)\gamma(f)$, where $f_o = \omega_o/2\pi$ is the cavity's resonant frequency, and $\gamma = I_2/I_1$ is the ratio of the output light flux to the input flux for monochramatic light with frequency $f = \omega_e/2\pi$ as derived and plotted in exercise 3.c.iii.

b. Show that the total energy flux $I_2^{\rm filtered}$ of the filtered, outcoming light is equal to $1/4\pi$ times the variance of the random process E_2, i.e. it is $(\Delta E_2)^2/4\pi$. (The 4π comes from electromagnetic theory.) Show further that

$$I_2^{\rm filtered} = \frac{S_{E_1}(f_o)\delta\omega_e/2\pi}{4\pi}\ , \tag{5}$$

where $\delta\omega_e$ is the half width of the cavity's resonance. (I hope I have the factors of 2 right!)

5. Contribution of Photon Shot Noise to the Total Gravity Wave Noise of an Idealized Interferometer. Consider the idealized gravitational-wave interferometer shown below:



A gravitational wave $h_+ \equiv h$, propagating vertically with + polarization, drives the motions of the end mirrors as shown, so that $\delta L(t) = \ell + \frac{1}{2}Lh(t)$, where 2ℓ is a constant, small difference in the lengths of the two cavities that is imposed for reasons we shall explore below. Assume that $|kBLh| \ll kB\ell \ll 1$. Assume that the timescale on which the wave and δL change is long compared to the light storage time in the interferometer arms. Then from exercise 3.d.iii we conclude that phase shifts $\pm\delta\phi = \pm 2Bk\delta L$ are put onto the light that emerges from the two arms of the interferometer. Here $\phi = \phi_o + \delta\phi$ where $\phi_o = 2Bk\ell$ is the static phase shift, and $\delta\phi = BkLh$ is the wave-induced phase shift.

a. Let I_o be the power (in Watts) of the incoming laser light impinging on the beam splitter. By analyzing the interaction of the light beams with the beam splitter, show that the intensity of the cavity-filtered light that travels from the beam splitter into the photo-detector is

$$I_{\rm PD}(t) = I_o[\phi_o + \delta\phi(t)]^2 \simeq I_o\phi_o^2 + 2I_o\phi_o\delta\phi(t) \ . \tag{6}$$

Denoting by $\bar{I}_{\rm PD} \equiv I_o\phi_o^2$ the time-averaged power into the photo-detector, show that Eq. (**??**) can be rewritten as

$$I_{\rm PD}(t) = \bar{I}_{\rm PD} + 2\sqrt{\bar{I}_{\rm PD} I_o} BkLh(t) \ . \tag{7}$$

[Note: The reason that we made the two arms slightly asymmetric (slight, opposite, offsets $\pm\ell$ of the two arm lenths from resonance) was to bring some steady light $I_o\phi_o^2$ out of the interferometer along with the signal light. The beating of the steady light against the signal light leads to an output light intensity (**??**) that is a linear function of $h(t)$. Without this slight asymmetry, the output light intensity would have been quadratic in $h(t)$ — which would not be nice. In real interferometers, other techniques (usually radio-frequency modulation and demodulation) are used to get a linear output signal.]

b. The steady part of the output light actually fluctuates slightly due to the randomness of the photon arrival times. The spectral density of these fluctuations is given by the standard shot-noise formula, which Kip derived in lecture 20:

$$S_{\bar{I}_{\rm PD}}(f) = 2\bar{I}_{\rm PD}\hbar\omega_e \ . \tag{8}$$

Show that this translates into the following noise for the gravitational wave signal that the experimenter infers from the measured output light power (**??**):

$$S_h(f) = \frac{\hbar\omega_e}{2I_o(BkL)^2} \ . \tag{9}$$

d. Note that this noise, like the shot noise itself, is independent of frequency; but the rms noise in a bandwidth equal to frequency,

$$h_{\rm rms} = \sqrt{fS_h(f)} = \sqrt{\frac{\hbar\omega_e f}{2I_o(BkL)^2}} \ . \tag{10}$$

increases as the square root of the frequency. Evaluate h_{rms} numerically for reasonable values of the parameters and compare it with numbers you have heard quoted for LIGO sensitivity.

c. When we allow the gravitational wave to vary on timescales comparable to or shorter than the light-storage time in the interferometer's arm cavity, we obtain a noise spectral density that is larger than the above result — larger by a factor $1 + f^2/f_o^2$, where $f_o = c/2\pi BL$ is the inverse of the mean time that light is stored in the interferometer arms. Explain why this amount of debilitation is reasonable. Draw a graph of the debilitated (i.e. the correct) gravitational-wave shot noise.