WEEK 16: LIGO's Facility Limits; Techniques for LIGO-III Interferometers and Beyond; Resonant-Mass ("Bar") Gravitational-Wave Detectors

Lecture 29 Part 1 by Kip [LIGO's Facility Limits]
Lecture 29, Part 2 by Ronald W.P. Drever [Techniques for LIGO-III and Beyond]
Lectures 30, by William O. Hamilton (LSU) [Resonant-Mass Detectors]

Reading Related to These Lectures:

Items in bold are recommended; others are supplementary.

A. LIGO Facility Limits

1. *Newtonian gravitational noise:*

   (a) *Gravitational noise produced by ambient ground motions* ("Seismic gravity gradient noise"): S. A. Hughes and K. S. Thorne, "Seismic Gravity-Gradient Noise in Interferometric Gravitational-Wave Detectors," *Physical Review D*, **58**, 122002 (1998); http://xxx.lanl.gov/abs/gr-qc/9806018 . **Section I is recommended;** the remainder is supplementary.

   (b) *Human gravitational noise:* K. S. Thorne and C. J. Winstein, "Human Gravity-Gradient Noise in Interferometric Gravitational-Wave Detectors," *Physical Review D*, **60**, 082001 (1999); http://xxx.lanl.gov/abs/gr-qc/9810016

2. *Light scattering noise in the LIGO beam tubes, and baffles to control it.* This is discussed in a series of internal LIGO documents written by Eanna Flanagan and Kip between 1994 and 1997, of which the following three are the most useful. These documents can be obtained via the web from LIGO Document Control, http://admdbsrv.ligo.caltech.edu/dcc/ ; search for them by document number or by author "Eanna Flanagan" or "K. Thorne".

   (a) E.E. Flanagan and K.S. Thorne, "Light Scattering and Baffle Configuration for LIGO," LIGO Technical Report LIGO-T950101-00-R, Caltech/MIT, January 1995. **Section III is recommended; it describes and quantifies the various processes that contribute to the scattered-light noise in the presence of baffles**.

   (b) E.E. Flanagan and K.S. Thorne, "Scattered Light Noise for LIGO," LIGO Technical Report LIGO-T950132-00-R, April 1995. Here the results of finite-element models of the beam-tube vibrations are combined with the light-scattering analyis of the previous reference to predict the spectral density of the light scattering noise.

   (c) E.E. Flanagan and K.S. Thorne, "Specifications for the Baffle Serrations in the LIGO Beam Tubes," LIGO Technical Report LIGO-T960012-00-R, Caltech/MIT, January 1996. This is the first document to spell out baffle serrations (teeth) with a Gaussian probability distribution. Nowhere in any of our documents is the theory behind the choice of the Gaussian distribution presented, except in words. Working out that theory is your task, in Exercise 3f below.

3. *Residual gas noise:* Peter R. Saulson, *Fundamentals of Interferometric Gravitational Wave Detectors* (World Scientific, 1994) — **Section 9.2, "Noise from Residual Gas"**. This material will be put on the course web site.

B. Techniques for LIGO-III Interferometers and Beyond.

4. K. Kuroda et. al., "Large-scale cryogentic gravitational wave telescope", *International Journal of Modern Physics D*, **8**, 557–579 (1999). This describes the plans for the Japanese "Large-scale Cryogenic Gravitation wave Telescope", LCGT, which Drever briefly discussed in his lecture. Note: Since this paper was written some of the values used in Figure 8 have changed and the Figure may not accurately represent later plans and data. Also, the LIGO-II noise curve shown in this paper is out of date.

5. R.W.P.Drever, "Concepts for Extending the Ultimate Sensitivity of Interferometric Gravitational Wave Detectors Using Non-Transmissive Optics with Diffractive or Holographic Coupling" Proc. of the Seventh Marcel Grossman Conference on General Relativity, Stanford, 1994. Eds. R.T. Jantzen and G. Mac Keiser (World Scientific, Singapore 1996), 1401-1406. This describes Drever's ideas about diffractive optics.

6. The following two references describe Drever's ideas and work on magnetic levitation of test masses:

   (a) R.W.P. Drever,"Techniques for Extending Interferometer Performance Using Magnetic Levitation and Other Methods" Proc. International Conference on Gravitational Waves, Sources and Detectors, Cascina (Pisa), Italy March 1996, Ed. I. Ciufolini and F. Fidecaro, (World Scientific 1997), 316-320.

   (b) R.W.P. Drever and S.J. Augst, Progress in Development of some Techniques Relating to Test-Mass Suspension and to the Extension of Interferometer Operation to Low Frequencies. Gravitational Wave Detection II. Proc. of the TAMA Workshop, Tokyo, October 1999, Eds. S. Kawamura and N. Mio (Universal Academy Press) 75-82.

7. R.W.P. Drever "Fabry-Perot Cavity Gravity-Wave Detectors" in "The Detection of Gravitational Waves" Ed. D.G. Blair, (Cambridge University Press, 1991) 306- 328. This describes some of Drever's other ideas.

C. Resonant-Mass Detectors.

8. David G. Blair, editor, *The Detection of Gravitational Waves* (Cambridge University Press, 1991). The following two chapters are a good introduction to resonant-mass detectors; they will be put on the course web site.

   (a) **David G. Blair, D.E. McClelland, H.-A. Bachor and R.J. Sandeman, Sections 3.2, 3.3, and 3.4 of Chapter 3, "Gravitational-wave detectors"**.

   (b) **David G. Blair, Chapter 4, "Resonant-bar detectors".**

9. Peter R. Saulson, *Fundamentals of Interferometric Gravitational Wave Detectors* (World Scientific, 1994), Chapter 4, "Resonant mass gravitational wave detectors." This is also a nice, pedagogical introduction.

10. **International Gravitational Event Collaboration (Z.A. Allen et. al.), "First search for gravitational wave bursts with a network of detectors", Physical Review Letters, 85, 5046–5050 (2000); http://lanl.arXiv.org/abs/astro-ph/0007308 .**

11. Web site of the International Gravitational Event Collaboration, IGEC: http://igec.lnl.infn.it/ .

EXERCISES

1. **Seismic Gravitational Noise**

   (a) A seismometer is placed on the ground near a LIGO test mass. It measures the spectral density $S_\xi(f)$ of the randomly oscillating vertical displacement $\xi(t) \equiv \delta z(t)$ of the ground produced by ambient seismic waves. The fluctuating mass distributions in the ground, caused by these ambient seismic waves, produce a fluctating gravitational force $F_x(t)$ on the test mass, along the horizontal direction $\mathbf{e}_x$ of the interferometer arm. The spectral density of this fluctuating force is $S_{F_x}(f)$. Use physical arguments and dimensional analysis to guess the "transfer function" from ground motions to the force on the test mass, i.e., guess the ratio $\sqrt{S_{F_x}/S_\xi}$, up to a dimensionless coefficient of order unity. It may be useful to notice that at the frequencies of interest (gravitational wave [GW] frequencies $f \sim 10$ to 100 Hz), the wavelength of the waves (~ 100 meters) is very large compared to the height of the test mass above the ground (~ 1 meter).

   (b) From your guess for $\sqrt{S_{F_x}/S_\xi}$, deduce the spectral density $S_x(f)$ of the test-mass motion (at GW frequencies); and — taking account of the fact that there are four test masses in the interferometer — deduce the spectral density $\sqrt{S_h(f)}$ of the seismic gravitational noise in the interferometer's GW output $h = \Delta L/L$. In the remainder of this exercise you will derive $\sqrt{S_h/S_\xi}$.

   (c) Consider a seismic wave propagating horizontally in the vicinity of a LIGO test mass. Idealize it as a plane, monochromatic wave with angular frequency $\omega = 2\pi f$ in the LIGO GW band, and let it propagate with horizontal phase speed c_H and in a direction that makes an angle θ with respect to the interferometer arm. This wave produces oscillatory changes of density that pull gravitationally on the test mass. These density changes are of two types: (i) the vertical rising and falling of the surface of the earth, which can be thought of as causing an oscillating "surface layer" of mass (positive mass when the ground moves up above its normal level, and negative mass when it moves down); and (ii) the density changes due to compression and expansion of soil or sand or clay beneath the earth's surface. It turns out that the "surface layer" contributes a large fraction of the gravitational pull on the test mass. Since it is the easiest to analyze and is independent of the details of the subsurface wave motion, we shall focus solely on it.

If ρ is the density of the material at the earth's surface and ξ is the vertical displacement of that surface material, what is the mass per unit area of the "surface layer" of mass produced by ξ?

(d) What is the gravitational force $F_x(t)$ exerted on the test mass by this oscillating surface layer? If you are feeling energetic, do the double integral to get the numerical coefficient of order unity; if you don't feel so energetic, just compute $F_x(t)$ up to the numerical coefficient.

(e) Now let a random superposition of seismic waves, propagating in all horizontal directions, exert a gravitational force on the test mass. Compute $\sqrt{S_{F_x}/S_\xi}$ [the transfer function you guessed in part (a)]. From this, compute $\sqrt{S_h(f)}$ caused by the Newtonian gravity of the seismic motions.

(f) Seismometer measurements at the LIGO sites reveal that

$$\begin{aligned}\sqrt{S_\xi(f)} &\sim 1\times 10^{-7}\frac{\text{cm}}{\sqrt{\text{Hz}}} \quad \text{at } 1 < f < 10\text{Hz}\ , \\ &\sim 1\times 10^{-7}\frac{\text{cm}}{\sqrt{\text{Hz}}}\left(\frac{10\text{Hz}}{f}\right)^2 \quad \text{at } f > 10\text{Hz}\ . \end{aligned} \tag{1}$$

Evaluate $\sqrt{S_h(f)}$ from seismic gravitational forces and compare it with the total LIGO-II noise $\sqrt{S_h(f)}$.

2. **Newtonian Gravitational Noise from Automobiles**

(a) Consider an automobile that moves with uniform speed v along a road that passes near a LIGO end test mass. For simplicity let the road run perpendicular to the interferometer arm. The passing automobile exerts a force F_x on the LIGO test mass.

i. Explain why the characteristic angular frequency of this force is $\omega_c \sim v/b$. Show that for physically reasonable values of v and b, this ω_c is small compared to the GW angular frequencies ω in LIGO's band of good sensitivity, $\omega_c \ll \omega$.

ii. Evaluate, to within a factor of order unity (or exactly if you wish) the Fourier transform $\tilde{F}_x(f)$ of the gravitational force that the passing automobile exerts on the test mass. Show that this force is "exponentially small", $\tilde{F}_x(f) \sim \exp(-\omega/\omega_c)$. This fact makes the gravity of uniformly moving objects typically be unimportant for LIGO. The dominant gravitational noise from moving objects is produced by sudden changes in their motion, and scales as some power of $1/\omega = 1/(2\pi f)$ rather than as an exponential. In the next part of this exercise we will explore an important example of this. The noise due to walking people, discussed by Kip in his Lecture 29 (and in Ref. [1b]) is another example.

(b) Consider an automobile that drives into the vicinity of an end test mass and parks. The parking involves a sudden stop, so there is a sudden change of acceleration Δa. For simplicity assume that the automobile parks at the end of the test mass's interferometer arm, facing the test mass, a distance b from it.

i. Compute the Fourier transform $\tilde{F}_x(f)$ of the gravitational force that the parking car exerts on the test mass. Focus solely on the influence of the sudden change of acceleration. Explain why this will be the automobile's dominant gravitational contribution to $\tilde{F}_x(f)$ in the LIGO frequency band.

ii. Show that the modulus of the Fourier transform of the gravitational-wave signal produced by the parking automobile is

$$|\tilde{h}(f)| = \alpha \frac{GM\Delta a}{Lb^3(2\pi f)^5} \; , \tag{2}$$

where M is the automobile's mass and α is a factor of order unity.

iii. For LIGO-II and for most any subsequent (LIGO-III or later) interferometer that might be operated in LIGO, the spectral density of the noise at low frequencies (below about 100 Hz) can be approximated as

$$\begin{aligned} S_h &= S_o(f_o/f)^n \quad \text{for } f > f_o \; , \\ &= \infty \quad \text{for } f < f_o \; . \end{aligned} \tag{3}$$

Here f_o is the frequency at which seismic noise (passing through the seismic isolation system) becomes important — i.e., it is the frequency of the seismic "wall". Compute the Signal to noise ratio produced in such an interferometer by the parking car, assuming that one uses optimal signal processing (matched filtering) to detect it; cf. Eq. (23) in Assignment 15.

iv. For LIGO-II $S_o = 4 \times 10^{-44}$/Hz, $f_o = 10$ Hz, and $n = 4$. Evaluate S/N for the parking car, assuming reasonable upper limits on M, and Δa. How far away from an end test mass must the parking lot be placed, to ensure that the gravitational force from the parking car will be undetectable.

v. A possible candidate for a LIGO-III interferometer is a "speedmeter QND interferometer" (Patricia Purdue and Yanbei Chen, paper in preparation), which has test masses with mass $m = 100$ kg, cryogenically cooled so that their thermal noises are negligible. This interferometer can beat the standard quantum limit, SQL, by a factor 4 in amplitude at all frequencies down to the seismic wall, which might be as low as in LIGO-III. Such an interferometer has the noise spectrum (3) with $f_o = 5$Hz, $n = 2$ and

$$S_o = \frac{1}{16} \frac{8\hbar}{m(2\pi f_o L)^2} \; , \tag{4}$$

where $\hbar$ is Planck's constant (divided by 2π) and $L = 4$ km is the interferometer arm length. For this LIGO-III interferometer, how far must the parking lot be from the LIGO test mass?

3. **Diffraction-Mediated Light Scattering Noise in LIGO**

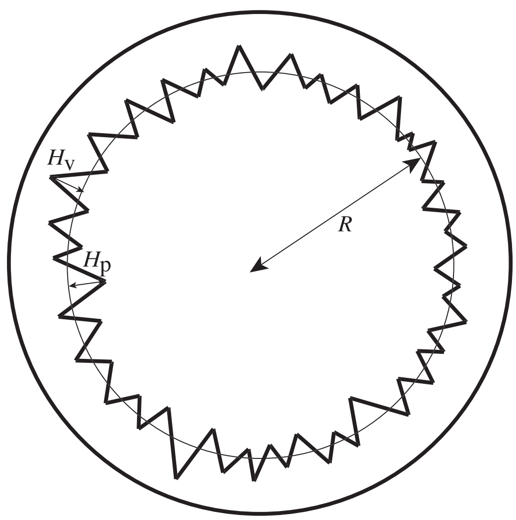
In this exercise you will learn why the baffles installed in the LIGO beam tubes to reduce scattered-light noise were endowed with serrations (teeth) whose peak heights and valley depths have Gaussian probability distributions.

The most dangerous situation for light scattering noise is when the mirrors of an arm cavity are at the centers of the beam tube's circular cross section, so the arm cavity's light beam is on the tube's central axis, and when the light scattering is axisymmetric. Then light scattered at different azimuthal angles ϕ can superpose coherently, causing a strong enhancement of the scattering noise, unless the baffles are designed to destroy coherence. The baffle teeth, by having random heights, achieve this coherence destruction.

In the dangerous case of axisymmetric scattering, the electic field of the light arriving at the baffle plane from the scattering mirror has the form

$$\psi_B \propto F(\theta)e^{ikr} \simeq F(\varpi/\ell_s)e^{ik\varpi^2/2\ell_s} \ . \tag{5}$$

Here $F(\theta)$ is proportional to the square root of the probability $\sqrt{dP/d\Omega(\theta)}$ for a photon to scatter into a unit solid angle at angle θ to the light beam, k is the wave vector, ℓ_s is the distance of the baffle plane from the scattering mirror, ϖ is cylindrical radius, and $r = \sqrt{\ell_s^2 + \varpi^2} \simeq \ell_s + \varpi^2/2\ell_s$ is distance propagated from the center of the scattering mirror.



Suppose that this scattered light encounters a baffle with the shape shown in the figure above. The mean radius of the baffle top is R, and it has teeth with peak heights H_p above the mean radius and valley depths H_v below the mean radius. (Both H_p and H_v are positive, by convention). Denote by $Y(\phi)$ the height of the baffle edge above the mean radius, at angular location ϕ; so $Y = H_p$ at a peak and $Y = -H_v$ at a valley; and so the radial location of the baffle top is $\varpi_B(\phi) = R - Y(\phi)$

The theory of Fresnel diffraction (based on "paraxial wave optics"; see, e.g., Chapter 7 of Blandford and Thorne) tells us that the electric field arriving at the other mirror can be computed by taking the light ψ_B at each point (ϖ, ϕ) in the baffle plane (in the region the baffle does not cover), propagating it to the mirror with the paraxial propagator $\propto e^{ikr} \simeq e^{ikl_r}e^{ik\varpi^2/2l_r}$, and superposing it coherently on the light from all the other points in the baffle plane. Here $\ell_r = L - \ell_s$ is the distance from the baffle plane to the "receiving mirror". In other words, the electric field arriving at the receiving mirror is

$$\psi_r = \int_0^{2\pi} d\phi \int_0^{\varpi_B(\phi)} d\varpi \; \psi_B e^{ik\varpi^2/2\ell_r} \; . \tag{6}$$

(a) Show that

$$\psi_r \propto F(R/l_s) \int_0^{2\pi} d\phi \int_0^{\varpi_B(\phi)} d\varpi \; e^{ik\varpi^2/2\ell} \; , \tag{7}$$

where $\ell \equiv \ell_s \ell_r / L$ and we approximate F as constant over the small radial variations between serration valleys and peaks.

(b) The ground's ambient seismic motions cause the baffles to vibrate. For simplicity idealize these vibrations as radial, so the radius of the baffle top varies with time as $\delta\varpi_B = \xi(t)$. These variations modulate the light arriving at the receiving mirror. Show that the modulated portion of that light has an electric field

$$\psi_{r,m} \propto \xi(t) \int_0^{2\pi} d\phi \exp\left[\frac{ik[\varpi_B(\phi)]^2}{2\ell}\right] \; . \tag{8}$$

(c) A small portion of this scattered light, upon hitting the receiving mirror, scatters back into the main beam. That portion which is out of phase with the main beam light produces a modulated phase shift of the mean beam, which can mimic the phase shift produced by a gravitational wave $h(t)$. Explain why the resulting gravitational wave noise is

$$h(t) \propto |\psi_{r,m}| \propto \xi(t) \left| \int_0^{2\pi} d\phi \exp\left[\frac{ik[\varpi_B(\phi)]^2}{2\ell}\right] \right| \; . \tag{9}$$

The goal of the baffle design is to choose the radius $\varpi_B(\phi)$ of the baffle top so as to make the integral as small as possible.

(d) The slopes of the baffle tooth edges (relative to the radial direction) are chosen all to be the same. Show that this implies

$$h(t) \propto \xi(t) \left| \sum_n e^{ikRH_{p,n}/2\ell} - \sum_n e^{-ikRH_{v,n}/2\ell} \right| \; , \tag{10}$$

where the sums are over the baffle peaks and valleys.

(e) One aspect of the baffle-design strategy is to choose the peak and valley heights such that the terms in Eq. (10) have random phases with respect to each other, so $h(t) \simeq \sqrt{2N}\xi(t)$ where N is the number of teeth (the number of terms in each sum); this corresponds to incoherent superposition of the light diffracted off the N teeth. If the phases in Eq. (10) were all the same, then we would have $h(t) \simeq 2N\xi(t)$, corresponding to coherent superposition of the light from the N teeth. To achieve the incoherence, it is necessary that the heights $H_{p,n}$ and $H_{v,n}$ be drawn from suitable random probability distributions for which the rms variations of the phases in Eq. (10) are about 2π or greater. Show that this will be so if the ranges of height variations for the peaks and for the valleys are $\pm\delta H$, where

$$\delta H \geq \pi\lambda L/R \simeq 2 \text{ millimeters} \; . \tag{11}$$

Here L is the arm length, λ is the wavelength of the light, and since the beam tube diameter is 1.2 meters, R is approximately 50 cm.

(f) Denote by $P_v(H_v)dH_v$ the probability that a baffle valley has depth H_v in the range dH_v and similarly by $P_p(H_p)dH_p$ the probability that a baffle peak has height H_p in the range dH_p. Show that the spectral density of the light scattering noise is

$$S_h(f) \propto S_\xi(f) N \left| \int P_p e^{ikRH_p/2\ell} dH_p - \int P_v e^{ikRH_v/2\ell} dH_v \right|^2 \tag{12}$$

This says the scattered light's amplitude noise is proportional to the Fourier transforms of the probability distributions for the baffle peaks and valleys. By choosing $P_p(H_p)$ and $P_v(H_v)$ to be Gaussian distributions, we can further reduce the noise below the "$\sqrt{N}$" level for incoherence. This is because the Fourier transform of a Gaussian is a Gaussian, and a Gaussian is very small for large values of its argument. The Gaussian distribution of heights does even better than simply breaking the coherence between baffle teeth: It causes a still stronger cancellation of the noise.

4. **Interaction of Gravitational Waves with a Resonant Mass Detector, and Thermal Noise**

Most resonant mass detectors are cylindrical bars with length large compared to diameter, and they are instrumented with transducers so as to monitor the vibrations of their fundamental longitudinal vibrations.

(a) Such a bar, so instrumented, can be idealized as two point masses separated by a spring. Each point mass has a mass equal to half that of the bar, $M/2$, and their separation is equal to the bar's length L. Orient the bar (spring) along the x direction, let a gravitational wave pass through the detector, traveling in the z direction, and denote by $\pm\xi(t)$ the displacements of the two idealized point masses from equilibrium. Explain why the equation of motion for these displacements is

$$\ddot{\xi} + \frac{\dot{\xi}}{\tau} + \Omega_o^2 \xi = a_{\text{fl}}(t) - \frac{L}{4}\ddot{h}_+(t) \ , \tag{13}$$

where $h_+(t)$ is the gravitational-wave field evaluated at the detector's center. Here $\Omega_o = 2\pi f_o$ (with $f_o \sim 1000$Hz) is the detector's angular eigenfrequency, τ is the damping time for its free oscillations due to coupling to other modes of the bar, and a_{fl} is the fluctional ("Langevin") acceleration caused by the coupling to those other modes — i.e., the detector's thermal noise.

This equation of motion neglects the coupling to a small probe mass (actually a diaphragm mounted on the end of the bar), which is used to amplify the bar's motion (as discussed in Hamilton's lectures). For simplicity we shall assume the bar has no such probe mass, so it has just one mechanical resonance rather than two.

(b) We can characterize the detector's motions by a complex amplitude $X \equiv X_1 + iX_2$ such that

$$\xi(t) = \Re\left[X e^{-i\Omega_o t}\right] = X_1(t)\cos\Omega_o t + X_2(t)\sin\Omega_o t \ . \tag{14}$$

This complex amplitude is often shown as a dot on an oscilloscope. In the absence of a gravitational wave, the complex amplitude $X = X_1 + iX_2$ changes slowly (on the long timescale τ) due to the fluctuational thermal noise and associated damping (the spot on the oscilloscope wanders). If one monitors this wandering over a time long compared to τ, what will be the rms magnitude $X_{\rm rms}$ of $|X| = \sqrt{X_1^2 + X_2^2}$?

(c) During a time Δt long compared to the eigenperiod $2\pi/\Omega_o$ but short compared to the damping time τ, what will be the typical magnitude $|\Delta X_T|$ of the thermally-induced change of X? [Hint: X_1 and X_2 undergo independent random walks.]

(d) In order to be detectable above the thermal noise, a gravitational wave must produce a change ΔX_h in the complex amplitude that is significantly larger than the typical thermally-induced change. If the wave acts for a time $\Delta t \ll \tau$ but $\Delta t \gg 2\pi/\Omega_o$ and is nearly sinusoidal with frequency as close as possible to the detector's eigenfrequency, how large must its amplitude be in order for the wave to be detected above the thermal noise?

5. **Readout, Noise, and Sensitivity of a Resonant Mass Detector**

(a) Turn, now, to the readout of the GW signal. Suppose the detector is so cold that its thermal noise is neglible, and that the noise in the readout system is also neglible. Explain why it then will be possible to infer all details of the gravitational waveform $h_+(t)$ from the motion $X(t)$ of the complex amplitude (from the moving spot on the oscilloscope).

(b) Because past technology has *not* permitted transducers to be strongly coupled to the bar until now (though that may change in the future), the readout system can achieve good sensitivity only by integrating up the transducer's signal over many eigenperiods of the bar's oscillations. This integrating up is possible because the Fourier components of the gravitational wave that are very nearly on resonance produce "permanent" changes in the complex amplitude X, i.e. changes that are preserved after the wave has passed, and are degraded by damping and thermal noise only on the long timescale τ. By contrast, Fourier componets well away from resonance produce only a temporary influence on $X(t)$, while the waves are driving the detector, and their influence therefore is not stored and cannot be integrated up. As a result, the detector's output noise $S_h(f)$ is large well away from resonance, and much quieter near resonance:

$$S_h(f) = S_o[1 + (f - f_o)^2/\delta f^2] \ . \tag{15}$$

Here, for Hamilton's Allegro detector, $\delta f \sim 1$ Hz and $S_o \sim 10^{-42}$/Hz; see Slide 15 of his second (CaJAGWR) lecture. Suppose that a broad-band gravitational-wave burst $h_+(t)$ interacts with the detector. Show that the signal-to-noise ratio is given by

$$\frac{S}{N} = \sqrt{\frac{4\pi|\tilde{h}(f_o)|^2\delta f}{S_o}} \ , \tag{16}$$

where $\tilde{h}(f_o)$ is the Fourier transform of $h_+(t)$ at the detector's resonant frequency. This explains why, in Ref. [10] the sensitivity of the ICEG gravitational-wave search is expressed in terms of the detectable value of $H \equiv |\tilde{h}(f_o)|$.

(c) Using the above noise values of S_o and δf, estimate the strength of the waves that are detectable by the ICEG network, and compare your result with the sensitivity reported in [10].

(d) When the strength of coupling of the transducer to the bar is changed, the bandwidth δf changes, but the ratio $S_o/\delta f$ does not change much (if one works hard to maximize the sensitivity). This means that the detectable wave strength $|\tilde{h}(f_o)|$ is nearly independent of the detector bandwidth δf. Why, then, are the experimenters working so hard to enlarge the detectors' bandwidths δf?

6. **Transducers for Resonant Mass Gravitational Wave Detectors**

In Joseph Weber's resonant-mass gravitational-wave detectors, the transducer that converted the bar's mechanical vibrations into electrical signals consisted of a set of piezo-electric crystals cemented around the middle of the bar as shown in the photograph below. When the bar vibrated, it squeezed each crystal, and the resulting strain in the crystal produced a voltage across it. (This is just the opposite of what is done in LIGO; there a voltage is applied to a crystal, causing its length to change, and that length change induces a phase shift in light passing through the crystal or reflected off a mirror attached to it. In Weber's case the crystal is used as a transducer from a mechanical signal to an electrical one; in LIGO's case, it is used as a transducer from an electrical voltage to a mechanical length change and thence to a light-beam phase change.)

In Weber's detector, the voltages of the squeezed crystals were added in series to get a large enough voltage for measurement. The measured total voltage was proportional to the strain, $\Delta L/L$ in Weber's bar, $V(t) \propto \Delta L(t)/L$. This is an example of a *passive* transducer: The voltage signal must rely on the energy of mechanical squeezing for its power.

Some modern resonant-mass detectors make use of *active* transducers, which Hamilton highlighted in his Lecture 30: The transducer consists of an electromagnetic *oscillator* with eigenfrequency ω_o and damping time τ_o (not to be confused with the bar's damping time τ). The oscillator is driven into large-amplitude oscillations by an electromagnetic *pump*, so its equation of motion is $\ddot{y} = -\omega_o^2 y - 2\dot{y}/\tau_o + De^{-i\omega_o t}$ with D the pumping amplitude, $y(t)$ one of the oscillators' electromagnetic quantities (its voltage or current or electric field at some location or ...), and τ_o the damping time when the oscillator's excitations are allowed to decay freely. It is easy to see that the driven, steady-state oscillations are given by $y = y_o e^{-i\omega_o t}$ where the amplitude $y_o = (\tau_o/2\omega_o)D$ is determined by a balance between the pump (characterized by D) and the dissipation (characterized by τ_o). To make this driven oscillator into a transducer, one arranges that the gravitational-wave-induced mechanical motion of some object (the *probe*) modulate the oscillator's eigenfrequency, so the oscillator's equation of motion is modified to

$$\ddot{y} = -\omega_o^2 \left(1 + [x(t)/H]\right) y - 2\dot{y}/\tau_o + De^{-i\omega_o t} \ . \tag{17}$$

Here $x(t)$ is the wave-induced displacement of the probe and H is some characteristic lengthscale that determines the coupling of the probe to the oscillator. This type of transducer is not only "active"; it is also a *parametric transducer*, which means that the mechanical motions couple to it by modulating its eigenfrequency.

(a) Assuming that x/H is extremely small (smaller than any other quantity of interest) as is always the case for gravitational-wave detectors, and that the GW timescale on which x changes is very long compared to the oscillator's eigenperiod, as is always the case for GW detectors, show that $x(t)$ produces a time-varying *phase shift* in the oscillator's excitations. A *readout* system monitors the oscillations' phase and thereby deduces $x(t)$ and thence the gravitational wave $h(t)$.

(b) In his first lecture, Hamilton highlighted a special case of such a parametric transducer: The case where $x(t)$ oscillates sinusoidally at twice the oscillator's eigenfrequency, $x = x_o \sin(2\omega_o t)$. (This is not the regime in which GW transducers generally operate, but it *is* the regime in which a child, pumping a swing, operates.) Show that in this case (which is called a "degenerate parametric amplifier"), the influence of x_o on the oscillator's excitation is amplified by a factor $\omega_o\tau_o$ relative to the strength of the excitation one gets in the slow-variation GW regime.

(c) Returning to GW detectors, for each of the following cases is the transducer *active* or *passive*; and if it is active, identify the *probe*, the *oscillator*, the *readout*, and the lengthscale H that characterizes the coupling of the probe to the oscillator: (i) The superconducting transducer used in Allegro, Hamilton's resonant-mass detector at LSU; see Hamilton's first lecture and also Reference [8b]. (ii) The microwave cavity transducer used in Niobe, the resonant-mass detector in Perth; see Reference [8b]. (iii) A LIGO interferometer. (Although the interferometer is not a resonant-mass detector, it can be discussed in the same language as we are using.)