Ph 237b: Gravitational Waves 1 May 2002

WEEK 15: GW Detection by Doppler Tracking of Spacecraft and Pulsar Timing; LIGO Data Analysis

Lecture 27 Part 1 by John Armstrong [GW Detection in LF Band by Doppler Tracking];
Lecture 27, Part 2 by Kip [GW Detection in HF Band by Pulsar Timing];
Lecture 28, by Albert Lazzarini [LIGO Data Analysis]

Reading Related to These Lectures:

Items in bold are recommended; others are supplementary. All these references (except [6]) are quite sophisticated and may be hard to follow without some preparation. The exercises are designed to provide that preparation: *It may be helpful to work the exercises before doing the reading!*

A. Doppler Tracking for GW Detection in the LF Band

1. Estabrook, F. B. and Wahlquist, H. D. *General Relativity and Gravitation*, **6**, 439 (1975). This classic paper derived the ”three pulse” response of a doppler tracking system to gravitational waves and explored some of the ”two-pulse” responses to specific noise sources.
2. Armstrong. J. W. *Radio Science*, **33**, 1727 (1998). This is the most recent published work on scintillation noise from the troposphere and plasma, in the LF gravitational-wave band.
3. J.A. Barnes, et. al., “Characterization of Frequency Stability,” *IEEE Trans. Instrum. Meas.*, Vol. **IM-20**, No. 2, pp. 105-120, May 1971. This is a classic paper about how to characterize noise in clocks.
4. Papers that report the details and results of GW searches via pulsar timing, from the most recent backward:
    (a) **Armstrong, J. W., Bertotti, B, Estabrook, F. B., Iess, L., and Wahlquist, H. D. ”The Galileo/Mars Observer/Ulysses Coincidence Experiment” in Proceedings of the Second Edoardo Amaldi Conference on Gravitational Waves (World Scientific: E. Coccia, G, Pizzella, G. Veneziano, eds), pp 159-167, 1998;** a copy is in the bookcase on the east wall of the Theoretical Astrophysics Interaction Room in Bridge Annex. [This is the only LF GW coincidence experiment ever performed: search for periodic and coalescing binary waveforms exploiting time-dependent transfer function; use of different s/c’s polarization-coupling to disqualify candidate signals]

    (b) Bertotti, B. et al. *Astronomy and Astrophysics* **296**, 13 (1995). Chirp and sinusoid search using data from the Ulysses spacecraft; limits to any massive companion orbiting the black hole at our galactic center

   (c) Anderson, J. D. Armstrong, J. W., and Lau, E. *Astrophysical Journal* **408**, 287 (1993). The first search/upper limits for chirps and coalescing binary waveforms in the LF band.

   (d) Armstrong, J. W., Estabrook, F. B. and Wahlquist, H. D. *Astrophysical Journal* **318**, 536 (1987) The first broadband search/upper limits for periodic GWs in the LF band.

   (e) Anderson, J. D. et al. *Nature,* **308**, 158 (1984) The first ”targeted” search in the LF band: no coherent GWs from Geminga at level claimed to be exciting solar oscillations.

   (f) Hellings, R. W. et al. *Phys Rev D*, **23**, 844 (1981). The first burst search in the LF band.

B. Pulsar Timing for GW Detection in the VLF Band.

5. Don Backer, transparencies from a March 3, 2001 CaJAGWR seminar on GW searches via pulsar timing: On the web at http://www.cco.caltech.edu/$\sim$cajagwr/pdf/backer.pdf

6. Roger Blandford, Ramesh Narayan and Roger W. Romani, “Arrival time analysis for a millisecond pulsar”, *Journal of Astrophysics and Astronomy*, **5**, 369–388 (1984). This is a readable early paper on the conceptual underpinnings for the use of pulsar-timing residuals to extract a variety of science results, including gravitational-wave signals, from pulsar-timing measurements.

7. Details and results of VLF GW searches via pulsar timing, from the most recent backward:

   (a) A.N. Lommen, “Precision multi-telescope timing of millisecond pulsars: New liimts on the gravitgational wave background and other restults from the pulsar timing array,” unpublished PhD thesis, University of California, Berkeley (Fall 2001). This is a broad overview of the pulsar timing work and plans of Don Backer’s UC Berkeley group and their collaborators. That collaboration is currently the leading effort, worldwide, on pulsar timing for GW detection. Among the results reported in Lommen’s thesis is a limit on stochastic background, $\Omega_{\rm GW} h^2 \lesssim 2 \times 10^{-9}$ at frequency $f \sim 1/17$yrs; see page 83 of the thesis. Here h is the Hubble expansion rate of the universe in units of 100 km/s/Mpc; current measurements suggest $h \simeq 0.7$.

   (b) **A.N. Lommen and D.C. Backer, “Using pulsars to detect massive black hole binaries via gravitational radiation: Sagittarius A* and nearby galaxies”, Astrophysical Journal, 562, 297–302 (2001) http://xxx.lanl.gov/abs/astro-ph/0107470.** This is the only published paper on results of the Backer group’s GW searches.

   (c) **V. Kaspi, J.H. Taylor and M.F. Ryba, “High-precision timing of millisecond pulsars. 3. long-term monitoring of PSRs B1885+09 and B1937+21”, Astrophysical Journal 428, 713–728 (1994).** [An electronic copy is available on the course website.] This is the highest-precision *published* search for a GW stochastic background via pulsar timing; the reported limit is $\Omega \lesssim 1 \times 10^{-8}$ at frequencies $f \sim 1/8$yrs.

   (d) D.R. Stinebring, M.F. Ryba and J.H. Taylor, “Cosmic gravitational-wave background: limits from millisecond pulsar timing”, *Physical Review Letters,* **65**, 285–288 (1990). This places a limit $\Omega_{\rm GW} h^2 \lesssim 10^{-7}$ at $f \sim 1/5$yrs.

8. M. Rajagopal and R.W. Romani, “Ultra-low frequency gravitational radiation from massive black hole binaries,” *Astrophysical Journal*, **446**, 543 (1005). This presents results of population synthesis calculations of the VLF stochastic background of GW’s from supermassive black-hole binaries

C. LIGO Data Analysis.

9. References on optimal signal processing as applied to gravitational-wave detectors
    (a) L.S. Finn, “Detection, measurement and gravitational radiation”, *Physical Review D*, **46**, 5236–5249 (1992); gr-qc/9209010. **Sections I, II and III are recommended reading.** This is a clearly written summary of the theory underlying both searches for waves and extraction of information from waves. Finn has extracted the basic ideas from the literature in other fields and brought it into the appropriate context and form for gravitational wave observations.
    (b) C. Cutler and E. Flanagan, “Gravitational waves from merging compact binaries: how accurately can one extract the binary’s parameters from the inspiral waveform?” it Phys. Rev. D **49**, 2658–2697 (1994). **Sections I and II are recommended reading.** This is a clearly written application of the above theory to waves from inspiraling binaries.
    (c) Extension of the above theory to simultaneous (joint) analysis of data from a network of detectors (such as the three LIGO detectors plus VIRGO and GEO600):
        i. Finn, gr-qc/0010033
        ii. S. Bose, gr-qc/0110041
    (d) The formulation of discrete template families for binary inspiral, and the counting of templates: B.J. Owen, **Phys. Rev. D 53**, 6749 (1996), and references therein.

10. Some textbooks dealing with data analysis that are cited in Lazzarini’s lecture:
    (a) An Introduction to Random Vibrations, Spectral & Wavelet Analysis, 3rd Ed.,D.E. Newland, Longman Singapore Publishers, Pte. Ltd. 1995 Engineering Applications of Correlation and Spectral Analysis, Bendat & Piersol, Wiley Interscience, 1980.
    (b) Mathematics in Science and Engineering, Vol 136, Dynamic System Identification: Experiment Design and Data Analysis, Goodwin & Payne, Academic Press, 1977.
    (c) Random Data, 2nd Ed., J. S. Bendat and A. G. Piersol, Wiley Interscience., 1986.
    (d) Probabilistic Random Variables and Stochastic Processes, by Papoulis 3rd Ed., McGraw-Hill.
    (e) Statistical Data Analysis, G. Cowan, Oxford Science Publication, Clarendon Press, 1998.
    (f) Data Analysis - A Bayesian Tutorial, D.S. Sivia, Oxford Science Publication, Clarendon Press, 1996.

11. Parallel computing and Amdahl's Law: Amdahl, G.M. Validity of the single-processor approach to achieving large scale computing capabilities. In AFIPS Conference Proceedings vol. 30 (Atlantic City, N.J., Apr. 18-20). AFIPS Press, Reston, Va., 1967, pp. 483-485.

12. Cleaning LIGO data via cross-channel regression: Allen, Hua, Ottewill (gr-qc/9909083).

13. Time-frequency data analysis techniques:

    (a) http://www-dsp.rice.edu/software/TFA
    (b) L. Cohen, Proc. of the IEEE, Vol 77, No. 7, July 1989
    (c) W. Anderson & R. Balasubramanian, PRD D60 102001 [This is the application of time-frequency techniques to LIGO data analysis]

14. Methods of searching for stochastic background:

    (a) P.F. Michelson, Mon. Not. Roy. Astron. Soc. 227, 933 (1987).
    (b) N. Christensen, Phys. Rev. D46, 5250 (1992)
    (c) E. Flanagan, Phys. Rev. D48, 2389 (1993), astro-ph9305029
    (d) B. Allen and J. Romano, Phys. Rev. D59, 102001 (1999), gr qc9710117

15. Searches for unmodeled gravitational-wave bursts using Flanagan's excess-power statistic:

    (a) W. Anderson et al., gr-qc/0001044
    (b) W. Anderson et al., gr-qc/0008066

16. Searches for waves from inspiraling binaries in data from the Caltech 40 meter prototype: B. Allen et al., gr-qc/9903108

17. The LIGO scientists are searching early LIGO-I data for waves from various types of sources; much of their ongoing work is documented on the web at the following sites:

    (a) Pulsars: http://www.lsc-group.phys.uwm.edu/pulgroup/
    (b) Binary insipral: http://www.lsc-group.phys.uwm.edu/iulgroup/
    (c) Stochastic background: http://feynman.utb.edu/ joe/research/stochastic/upperlimits/
    (d) Unmodeled bursts: http://www.ligo.caltech.edu/ ajw/bursts/bursts.html

EXERCISES

1. **Alan Variance** [Exercise 5.4 in Blandford and Thorne, Chapter 5:]

   Highly stable clocks (e.g., Rubidium clocks or Hydrogen maser clocks) have angular frequencies ω of ticking which tend to wander so much over long time scales that their variances are divergent. More specifically, they typically show flicker noise on long time scales (low frequencies)

$$S_\omega(f) \propto 1/f \quad \text{at low } f \; ; \tag{1}$$

and correspondingly,

$$\sigma_\omega{}^2 = \int_0^\infty S_\omega(f) df = \infty \ . \tag{2}$$

For this reason and others, clock makers have introduced a special technique for quantifying the frequency fluctuations of their clocks: They define

$$\phi(t) = \int_0^t \omega(t')dt' = (\text{phase}) \ , \tag{3}$$

$$\Phi_\tau(t) = \frac{[\phi(t+2\tau) - \phi(t+\tau)] - [\phi(t+\tau) - \phi(t)]}{\sqrt{2}\bar{\omega}\tau} \ , \tag{4}$$

where $\bar{\omega}$ is the mean frequency. Aside from the $\sqrt{2}$, this is the fractional difference of clock readings for two successive intervals of duration τ. [In practice the measurement of t is made by a clock more accurate than the one being studied; or, if a more accurate clock is not available, by a clock or ensemble of clocks of the same type as is being studied.]

(a) Show that the spectral density of $\Phi_\tau(t)$ is related to that of $\omega(t)$ by

$$\begin{aligned} S_{\Phi_\tau}(f) &= \frac{2}{\omega^2}\left[\frac{\cos 2\pi f\tau - 1}{2\pi f\tau}\right]^2 S_\omega(f) \\ &\propto f^2 S_\omega(f) \ \text{ at } f \ll 1/2\pi\tau \ , \\ &\propto f^{-2} S_\omega(f) \ \text{ at } f \gg 1/2\pi\tau \ . \end{aligned} \tag{5}$$

Note that $S_{\Phi_\tau}(f)$ is much better behaved (more strongly convergent when integrated) than $S_\omega(f)$, both at low frequencies and at high.

(b) The *Alan variance* of the clock is defined as

$$\sigma_\tau{}^2 \equiv [\text{ variance of } \Phi_\tau(t)] = \int_0^\infty S_{\Phi_\tau}(f) df \ . \tag{6}$$

Show that

$$\sigma_\tau = \left[\alpha \frac{S_\omega(1/2\tau)}{\bar{\omega}^2} \frac{1}{2\tau}\right]^{\frac{1}{2}} \ , \tag{7}$$

where α is a constant of order unity which depends on the spectral shape of $S_\omega(f)$ near $f = 1/2\tau$.

(c) Show that if ω has a white-noise spectrum (S_ω independent of f), then the clock stability is better for long averaging times than for short [$\sigma_\tau \propto 1/\sqrt{\tau}$]; that if ω has a flicker-noise spectrum ($S_\omega \propto 1/f$), then the clock stability is independent of averaging time; and if ω has a random-walk spectrum ($S_\omega \propto 1/f^2$), then the clock stability is better for short averaging times than for long.

2. **Gravitational-wave signals in pulsar timing data**

Consider a plane gravitational wave with $+$ polarization propagating in the z-direction, as described in TT coordinates:

$$ds^2 = -dt^2 + [1 + h(t-z)]dx^2 + [1 - h(t-z)]dy^2 + dz^2 \ . \tag{8}$$

From exercise 4 of Week 6 we know that a freely moving object initially at rest in the TT coordinate system remains always at rest. Suppose that a pulsar is at rest at the spatial origin, and the earth is at rest at spatial location $x = \alpha L$, $y = \beta L$, $z = \gamma L$, where $\alpha^2 + \beta^2 + \gamma^2 = 1$. Note from the metric (8) that proper time, as measured by the pulsar, is equal to coordinate time t, and similarly for the earth.

(a) Radio waves are emitted toward Earth by the pulsar at its proper times $t = 0,\ \tau,\ 2\tau,\ ...$ — i.e., with precisely constant interpulse spacings τ. Show that, to first order in h, the proper time $t_{\rm rec}$ that a pulse is received at earth differs from the proper time $t_{\rm em}$ it is emitted at the pulsar by

$$t_{\rm rec} - t_{\rm em} = \frac{\alpha^2 - \beta^2}{2} \int_0^L h(t_{\rm rec} - L + [1-\gamma]r)dr \ , \tag{9}$$

where the speed of light has been set to unity and the integral is with respect to distance traveled by the pulse, from pulsar to earth.

[Hint: the radio waves propagate along null rays, i.e. null geodesics. There is an action principle for geodesics; the action is

$$I = \int g_{\alpha\beta} \frac{dx^\alpha}{d\zeta} \frac{dx^\beta}{d\zeta} d\zeta \ , \tag{10}$$

where the integral is along the geodesic $x^\alpha(\zeta)$ which is parametrized by an "affine parameter" ζ, and the endpoints are held fixed. If you are not familiar with this action principle, you might wish to vary it with respect to the world line $x^\mu(\zeta)$ with endpoints held fixed and verify that the resulting Euler-Lagrange equations are equivalent to the geodesic equation. Apply this action principle to the ray along which the pulse travels. The value of the action must be zero, but we don't know the precise form of the ray's geodesic world line $x^\mu(\zeta)$ without solving the Euler Lagrange (geodesic) equations. However, we do know that if we use a world line that is slightly wrong (by a fractional amount of order h), we will get an action that differs from zero by an amount of order h^2, which is negligible. Thus, we can use as the world line along which to evaluate the action $(t, x, y, z) = (1, \alpha, \beta, \gamma)\zeta$ with $\zeta = r =$(distance traveled). (This would be the ray's geodesic world line in the absence of the wave, so it must differ from the correct geodesic world line in the presence of the wave by an amount of order h.)]

(b) Denote by $R(t)$ the "residual" in pulsar timing measurements — i.e., the amount by which the arrival time of a pulse $t_{\rm rec}$ differs from what one expects based on all one's passed timing data and on one's best model for the dynamics (rotation, orbital motion, ...) of the pulsar. The expression (9) is the contribution of the gravitational wave to the residual. Suppose the wave is more or less sinusoidal with characteristic frequency f_o and with amplitude h_o. Show that to within a factor of order unity, the wave-induced residual is

$$R \sim \frac{h_o}{4\pi f} \sim 3\mu{\rm sec}\frac{h_o}{10^{-13}} \frac{1/10{\rm yr}}{f_o} \ . \tag{11}$$

(The current state of the art in pulsar timing is an ability to measure residuals, over timescales of 20 years, to a precision of order a microsecond.)

(c) Suppose there is a gravitational wave background with energy density, in units of that required to close the universe, $\Omega_{\rm GW}$. Show that these waves produce rms fluctuations of the residual, in a bandwidth equal to frequency, given by

$$R_{\rm rms} \sim \sqrt{\frac{\Omega_{\rm GW}}{10^{-8}} \left(\frac{1/10{\rm yr}}{f}\right)^4} \,\mu{\rm sec}\ . \qquad (12)$$

3. **Gravitational-wave signals in doppler tracking of spacecraft**

Now place the earth at rest at the spatial origin and a spacecraft at rest at $x = \alpha L$, $y = \beta L$, $z = \gamma L$, where $\alpha^2 + \beta^2 + \gamma^2 = 1$, and let the plane-fronted gravitational wave (8) pass through the solar system.

(a) A coded microwave signal is sent out from Earth to the spacecraft, where it is immediately transponded back to Earth. Derive a formula for the influence of the gravitational wave on the round-trip travel time, $t_{\rm rec} - t_{\rm em}$ of a bit of the code. Your formula should be a simple generalization of the pulsar-timing formula (9). As with that formula, express your answer as a function of the time of receipt of the bit of code, $t_{\rm rec}$. This method of tracking a spacecraft and searching for gravitational waves is called *ranging*.

(b) It turns out that one can get better sensitivity, in a gravitational-wave search, by measuring the fractional frequency shift $Z = \Delta\nu/\nu \equiv (\nu_{\rm rec} - \nu_{\rm em})/\nu_{\rm em}$ of the microwaves than by measuring their round-trip travel time. Spacecraft tracking by means of the fractional frequency shift is called *doppler tracking*, since the principal contribution to Z is the doppler shift due to the motions of the Earth and the spacecraft (motions that we have ignored in our simple version of the analysis). By differentiating your formula for the ranging signal, show that the influence of the gravitational waves on the doppler signal is given by

$$Z(t) = \frac{1}{2}\left(\frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2}\right)\Big[(1+\gamma)h(t-2L) - 2\gamma h(t - [1+\gamma]L) - (1-\gamma)h(t)\Big]\ , \qquad (13)$$

where t is the arrival time of the signal, denoted $t_{\rm rec}$ above. This is the three-pulse answer originally derived in the classic paper by Estabrook and Wahlquist, Ref. [9]). Interpret each of the three pulses in terms of the interaction of the gravitational waves with the earth or the spacecraft.

4. **Gravitational-wave signals in LISA and LIGO**

Now place a corner drag-free spacecraft of LISA at the spatial origin, one end spacecraft at $x = \alpha L$, $y = \beta L$, $z = \gamma L$, and another end spacecraft at $x = \alpha' L$, $y = \beta' L$, $z = \gamma' L$. Laser light is sent out from the corner spacecraft to the two end spacecraft which transpond the light back phase coherently, and the two returning light beams are then interfered so as to deduce their difference in phase. The interference of these two beams is one of many LISA data channels, as we shall learn later in this course.

(a) What is the gravity-wave-induced difference in phase of the returning beams? [You should be able to derive your answer, quickly, from your answer for the range signal between earth and one spacecraft, Exercise 3a.]

(b) For LIGO (but not generally for LISA) the wavelengths of the gravitational waves are large compared to the separation L between corner and end stations. In your answer for the phase shift in LISA, take a long-wavelength limit and thereby infer the difference in phase shift between the two arms of LIGO, in the idealized case where there are no corner test masses — i.e., where there is just a beam splitter and two end test masses.

(c) Specialize to the case where the waves come in from overhead and have polarization axes aligned with LIGO's arms ($\alpha = 1, \beta = 0, \gamma = 0; \alpha' = 0, \beta' = 1, \gamma' = 0$). Then your answer for the difference in phase shift should agree with the standard LIGO answer,

$$\Delta\phi(t) = 2\omega_o L h(t) \ . \tag{14}$$

5. **Wiener's Optimal Filter** [Exercise 5.3 of Blandford and Thorne, Chapter 5]

Suppose that you have a gravitational-wave detector with noise spectral density $S_h(f)$, and that you are expecting a gravitational-wave signal $s(t)$ with finite duration and a known waveform to come in, beginning at a predetermined time $t = 0$, but you are not sure whether it is present or not. (In the next exercise we will deal with the more realistic case of a signal with unknown arrival time and at least partially unknown waveform.) If the signal is present, then your detector's output will be

$$h(t) = s(t) + n(t) \ , \tag{15}$$

where $n(t)$ is the noise, a random process with the spectral density density $S_h(f)$ and with zero mean, $\bar{n} = 0$. If the signal is absent, then the output is $h(t) = n(t)$. A powerful way to find out whether the signal is present or not is by passing the actual output $h(t)$ through a filter with a carefully chosen kernel $K(t)$. More specifically, compute the number

$$W \equiv \int_{-\infty}^{+\infty} K(t) h(t) dt \ . \tag{16}$$

If $K(t)$ is chosen optimally, then W will be maximally sensitive to the signal $s(t)$ and minimally sensitive to the noise $n(t)$; and correspondingly, if W is large you will infer that the signal was present, and if it is small you will infer that the signal was absent. This exercise derives the form of the *optimal filter*, $K(t)$, i.e., the filter that will most effectively discern whether the signal is present or not. As tools in the derivation we use the quantities S and N defined by

$$S \equiv \int_{-\infty}^{+\infty} K(t) s(t) dt \ , \quad N \equiv \int_{-\infty}^{+\infty} K(t) n(t) dt \ . \tag{17}$$

Note that S is the filtered signal, N is the filtered noise, and $W = S + N$. Since $K(t)$ and $s(t)$ are precisely defined functions, S is a number; but since $n(t)$ is a random process, the value of N is not predictable and instead is given by some probability distribution $p(N)$. We shall also need the Fourier transform $\tilde{K}(f)$ of the kernel $K(t)$.

(a) In the measurement being done, one is not filtering a function of time to get a new function of time; rather, one is just computing a number, $W = S + N$. Nevertheless, as an aid in deriving the optimal filter it is helpful to consider the time-dependent output of the filter which results when noise $n(t)$ is fed continuously into it:

$$N(t) \equiv \int_{-\infty}^{+\infty} K(t'-t)n(t')dt' \; . \tag{18}$$

Show that this random process has a mean squared value

$$\overline{N^2} = \int_0^{\infty} |\tilde{K}(f)|^2 S_h(f) df \; . \tag{19}$$

Explain why this quantity is equal to the average of the *number* N^2 computed via (17) in an ensemble of many experiments:

$$\overline{N^2} = \langle N^2 \rangle \equiv \int p_1(N) N^2 dN = \int_0^{\infty} |\tilde{K}(f)|^2 S_h(f) df \; . \tag{20}$$

We shall denote this mean-square output noise by $\mathcal{N}^2 \equiv \langle N^2 \rangle$.

(b) Show that of all choices of $K(t)$, the one that will give the largest value of the "signal-to-noise ratio"

$$\frac{S}{\mathcal{N}} \tag{21}$$

is Norbert Wiener's (1949) optimal filter: the $K(t)$ whose Fourier transform $\tilde{K}(f)$ is given by

$$\tilde{K}(f) = \text{const} \times \frac{\tilde{s}(f)}{S_h(f)} \; , \tag{22}$$

where $\tilde{s}(f)$ is the Fourier transform of the signal $s(t)$ and $S_h(f)$ is the spectral density of the noise. Note that when the noise is white, so $S_h(f)$ is independent of f, this optimal filter function is just $K(t) = \text{const} \times s(t)$; i.e., one should simply multiply the known signal form into the receiver's output and integrate. On the other hand, when the noise is not white, the optimal filter (22) is a distortion of $\text{const} \times s(t)$ in which frequency components where the noise is large are suppressed, while frequency components where the noise is small are enhanced.

(c) Show that, independently of how the constant in Eq. (22) is chosen, the squared signal to noise ratio, using the optimal filter, is

$$\frac{S^2}{\langle N^2 \rangle} = 4 \int_0^{\infty} \frac{|\tilde{s}(f)|^2}{S_h(f)} df \; . \tag{23}$$

This motivates a common definition of an inner product on waveform space: If $p(t)$ and $q(t)$ are two waveforms for gravitational waves that interact with a detector whose noise spectral density is $S_h(f)$, then the inner product of those two waveforms is

$$\langle p|q \rangle \equiv 2 \int_0^{\infty} \frac{\tilde{p}\tilde{q}^* + \tilde{p}^*\tilde{q}}{S_h} df \; . \tag{24}$$

The squared signal to noise ratio (23) is then the inner product of the signal with itself, $S/\mathcal{N} = \langle s|s \rangle$. Much of the theory of signal processing, as summarized in Lazzarini's lecture and presented in References 9, is based on the inner product (24).

6. **Searching for an Inspiraling Binary Signal By the Method of Matched Filters**

Consider a search for gravitational waves from an inspiraling neutron-star or black-hole binary using a detector with noise spectral density $S_h(f)$. Denote by $h(t)$ the output of the detector. Idealize the binary's waveform as that predicted by the Newtonian, quadrupole approximation [Eqs. (17) of Week 5 with the frequency evolving with time in the manner deduced in Exercise 4 of Week 6]. The Fourier transform of this waveform, when one allows for an unknown time t_o for the endpoint of the inspiral and an unkown gravitational-wave phase ϕ_o at the endpoint, is

$$\tilde{s}(\mathcal{M}, t_o, \phi_o; f) = \exp[i(2\pi f t_o - \phi_o)]\tilde{s}_0(\mathcal{M}, f) \quad \text{for } f \geq 0 \ , \tag{25}$$

where

$$\tilde{s}_0(\mathcal{M}, f) = \alpha f^{-7/6} \exp\left[i\frac{3}{4}(8\pi\mathcal{M}f)^{-5/3} - i\frac{\pi}{4}\right] \quad \text{for } f \geq 0 \tag{26}$$

is $\tilde{s}(f)$ for $t_o = \phi_o = 0$, α is a constant, and $\mathcal{M}$ is the binary's unknown chirp mass. Since $s(t)$ is real, $\tilde{s}(-f) = \tilde{s}(f)^*$.

Not surprisingly, the optimal way to search for this binary's waves is to use the Wiener optimal filter with a Kernel constructed from the signal $\tilde{s}(\mathcal{M}', t_o', \phi_o'; f)$ for *trial* values of the chirp mass, endpoint time and endpoint phase

$$\tilde{K}(\mathcal{M}', t_o', \phi_o', f) = \frac{\tilde{s}(\mathcal{M}', t_o', \phi_o'; f)}{S_h(f)} \ , \tag{27}$$

and then maximize the filter output over the trial values of the parameters $\mathcal{M}', t_o', \phi_o'$. [The waveforms $s(\mathcal{M}', t_o', \phi_o'; t)$ and their Fourier transforms $\tilde{s}(\mathcal{M}', t_o', \phi_o'; f)$ used to construct the filters K are called *templates*, and this method of search is called the *method of matched filters*.]

(a) In performing such a search, one must be careful about the value of the constant α in the optimal filter [Eqs. (27), (26), (25)], since one is comparing computed values of the filtered output for different filters. Explain why one should choose α in such a manner that the rms filtered noise $\mathcal{N}$ is the same for the trial templates. Show that this is equivalent to asking that all the trial templates $s(t)$ have the same inner product with themselves. It is conventional to choose that inner product to be unity. In other words, in the optimal filters we must choose the constant α in such a way as to guarantee that

$$\langle s|s\rangle = 1 \tag{28}$$

for every template. Show that this normalization condition means that α is a certain function of the trial chirp mass $\mathcal{M}'$, and derive that function.

(b) Show that the above prescription for the binary search reduces to computing

$$W = \max_{\phi_o', t_o', \mathcal{M}'} \Re\left[e^{-i\phi_o'} \int_0^\infty e^{i2\pi f t_o'} \ \frac{\tilde{s}_0(\mathcal{M}', f)}{S_h(f)} df\right] \ . \tag{29}$$

(c) Show further that the maximization over the unknown final phase ϕ'_o can be done analytically and gives

$$W = \max_{t'_o, \mathcal{M}'} \left| \int_0^\infty e^{i2\pi f t'_o} \, \frac{\tilde{s}_0(\mathcal{M}', f)}{S_h(f)} df \right| \; . \tag{30}$$

Note that the integral is a Fourier transform. Such Fourier transforms can be done very quickly on a computer, giving all at once the values for a huge discrete sequence of trial ending times t'_o. The two parameters ϕ'_o and t'_o, which can be dealt with quickly and easily in the data analysis, are called *extrinsic parameters*. The remaining, *intrinsic parameter* $\mathcal{M}'$ determines the *shape* of the template. Its trial values must be dealt with one by one, with separate Fourier transforms.

(d) The inspiral templates have a large number of cycles with gradually changing frequency; and when one uses more realistic waveforms as templates, with post-Newtonian corrections and modulations due to spin-orbit and spin-spin coupling, the templates become quite complex and (presumably) very different from spurious noises in the interferometers. As a result, it is reasonable to expect that, in the absence of any real signal, the probability distribution for the output of the filter before maximization [i.e. for the quantity N being maximized in Eq. (29)] will be a Gaussian, with mean zero and rms value $\mathcal{N}$,

$$P_G(N) = \frac{e^{-N^2/2\mathcal{N}^2}}{\sqrt{2\pi\mathcal{N}^2}} \; ; \tag{31}$$

cf. the "central limit theorem" of probability theory [e.g., Sec. 5.2 of Blandford and Thorne, Chapter 5]. Show that, after performing the maximization over the trial phase ϕ'_o, the probability distribution for the filter output [the quantity being maximized in Eq. (30)] becomes a Rayleigh distribution

$$P_R(N) = \frac{N e^{-N^2/2\mathcal{N}^2}}{\mathcal{N}^2} \; . \tag{32}$$

(e) The noise curves for the LIGO interferometers have noise minima at $f \sim 100$ Hz, and $S_h(f)$ grows above that minimum by an order of magnitude at $f \sim 1000$ Hz. This means that the best achievable resolution for the ending time t'_o is $\Delta t'_o \sim 1/1000\text{Hz} = 1$ millisecond. In a search lasting one year, then, one must explore of order 3×10^{10} values of t'_o. It turns out one must also explore of order 3000 values of the chirp mass $\mathcal{M}'$. This means that, in effect, in Eq. (30) one must maximize over $\sim 10^{14}$ more or less independent values of the filter output. How high must the signal to noise $S/\mathcal{N}$ of this maximum be in order to have a *false alarm* probability (probability of a spurious detection) less than 10^{-3}? Suppose (as will be the case) that one demands coincidence in at least two 4-km LIGO interferometers. Then how high must $S/\mathcal{N}$ be in each interferometer for a false alarm probability less than 10^{-3}? Suppose one has also the 3-km VIRGO interferometer and that all three big interferometers have nearly the same noise curves. Then how high must $S/\mathcal{N}$ in each interferometer be?