Ph 237b: Gravitational Waves 24 April 2002

WEEK 14: LIGO as a "Large Science" Project;
Quantum Optical Noise in Advanced LIGO Interferometers

Lecture 25 by Barry Barish [LIGO-II as a "Large Science Project"];
Lecture 26, by Alessandra Buonanno and Yanbei Chen [Quantum Optical Noise in Advanced LIGO]

Reading Related to These Lectures:

Items in bold are recommended; others are supplementary. All these references (except [6]) are quite sophisticated and may be hard to follow without some preparation. The exercises are designed to provide that preparation: *It may be helpful to work the exercises before doing the reading!*

A. References relevant to the Buonanno-Chen lecture: Quantum optical noise and beating the standard quantum limit

1. **A. Buonanno and Y. Chen, "Optical noise correlations and beating the standard quantum limit in advanced gravitational wave detectors", Classical and Quantum Gravity. 18, L95–L101 (2001).** Available on the web at http://www.iop.org/EJ/S/1/NCA143559/ , and at http://lanl.arXiv.org/abs/gr-qc/0010011 .

2. **A. Buonanno and Y. Chen, "Laser-interferometer gravitational-wave optical-spring detectors", Classical and Quantum Gravity, 19, 1569–1574 (2002).** Available on the web at http://www.iop.org/EJ/S/1/NCA143559/ , or at http://lanl.arXiv.org/abs/gr-qc/0201063 .

3. A. Buonanno and Y. Chen, "Quantum noise in second generation, signal-recycled laser interferometric gravitational-wave detectors", *Physical Review D*, **64**, 042006 (2001); especially Sections I, II, III. Available on the web at http://prd.aps.org/ , and at http://lanl.arXiv.org/abs/gr-qc/0102012 .

4. A. Buonanno and Y. Chen, "Signal recycled laser-interferomter gravitational-wave detectors as optical springs", *Physical Review D*, **65**, 042001 (2002); especially Sections I, III, IV. Available on the web at http://prd.aps.org/ , and at http://lanl.arXiv.org/abs/gr-qc/0107021 .

5. H.J. Kimble, Yu. Levin, A. Matsko, K.S. Thorne and S. Vyatchanin, "Conversion of conventional gravitational-wave interferometers into quantum nondemolition interferometers by modifying their input and/or output optics", *Physical Review D* **65**, 022002 (2001); Sections I, II, III, IV.A, and IV.C. Available on the web at http://prd.aps.org/ , and at http://lanl.arXiv.org/abs/gr-qc/0008026 .

6. R.D. Blandford and K.S. Thorne, *Applications of Classical Physics*, **Section 5.3 of Chapter 5**; available on the web at http://www.pma.caltech.edu/Courses/ph136/ph136.html .

B. References relevant to Barish's lecture: The scientific management of large projects.

7. Jain, R.K. and Triandis, H.C. *Management of Research and Development Organizations: Managing the Unmanageable* (Second Edition), John Wiley & Sons, Inc., 1997, ISBN 0-471-14613

8. Sapienza, A.M., *Managing Scientists: Leadership Strategies in Research and Development*, John Wiley & Sons, Inc., 1995, ISBN 0-471-04367-2

9. Dinsmore, P.C., *Human Factors in Project Management* (Revised Edition), American Management Association

10. Lewis, J.P., *The Project Manager's Desk Reference: A Comprehensice Guide to Project Planning, Scheduling, Evaluation, and Systems* (Second Edition), McGraw-Hill

11. Traweek, S., *Beamtimes and Lifetimes: The World of High-Energy Physicists*, Harvard University Press, 1988, ISBN 0-674-06347-3.

12. *National Collaboratories: Applying Information Technology for Scientific Research, Computer Science and Telecommunications*, Board of the National Academy of Sciences, 118 pages, 1993, ISBN 0-309-04848-6, Library of Congress Catalog #93-083795.

EXERCISES

Note: Exercise 1 provides an elementary introduction to some of the key ideas and notation used by Buonanno and Chen. Exercises 2–5, devised by Buonanno and Chen (with minor changes by Kip) provide insights into the Buonanno-Chen lecture and reading. These exercises do *not* involve extensive or difficult calculations. Rather, they are designed for pedagogy.

1. **Shot noise as a beating of vacuum fluctuations against classical light**

   Consider a beam of monochromatic light propagating in the $+z$ direction. The beam has some cross-sectional profile (typically Gaussian) which does not interest us in this problem set, so we shall idealize it as a plane wave with a finite cross-sectional area $\mathcal{A}$ that is constant. The beam's electric field then has the following form

$$E = [D + E_1^{\rm vac}(t - z/c)]\cos[\omega_0(t - z/c)] + E_2^{\rm vac}(t - z/c)\sin[\omega_0(t - z/c)]\,. \qquad (1)$$

   Here D is a real number (the amplitude of the light), c is the speed of light, and $E_{1,2}^{\rm vac}(t - z/c)$ are fluctuations forced onto the light by the laws of quantum mechanics; $E_1^{\rm vac}$ is the fluctuating electric field amplitude associated with the beam's cosine quadrature (the quadrature in which all the light except its fluctuations resides), and $E_2^{\rm vac}$ is that associated with the sine quadrature. Because the fluctuations $E_{1,2}^{\rm vac}$ remain even when D is set to zero, they are called *vacuum fluctuations*. It is conventional to introduce dimensionless vacuum fluctuations $a_{1,2}(t - z/c)$ related to the electric-field vacuum fluctuations by

$$E_{1,2}^{\rm vac}(t - z/c) = \sqrt{\frac{4\pi\hbar\omega_0}{\mathcal{A}c}}\,a_{1,2}(t - z/c)\ . \qquad (2)$$

Here $\hbar$ is Planck's constant (divided by 2π), ω_0 is the light beam's monochromatic frequency, and $\mathcal{A}$ is its cross sectional area. In quantum electrodynamics, $E_{1,2}^{\rm vac}(t - z/c)$ and $a_{1,2}(t - z/c)$ are Hermitian operators (observables); but we can equally well regard them as classical random processes whose spectral densities are deducible from the laws of quantum electrodynamics and depend on the light's quantum state. We shall adopt this (semiclassical) viewpoint throughout almost all of this problem set.

If the light is as quiet as quantum mechanics allows, then these operators are in the standard vacuum state (1/2 quantum of fluctuation in each mode), and the light [Eq. (1)], with its nonzero classical amplitude D, is said to be in a *coherent state*. In other words, a coherent state of light is a superposition of classical (monochromatic) light and standard vacuum fluctuations. This is the type of light that an ideally quiet laser produces, and the type that LIGO-I scientists try to achieve in their interferometers. [In some designs for advanced interferometers, the light is in some other quantum state — e.g., a *squeezed state*, which is a superposition of the classical light beam D on "squeezed vacuum fluctuations", for which $E_{1,2}^{\rm vac}(t - z/c)$ and $a_{1,2}(t - z/c)$ are in a so-called squeezed vacuum state.] In this problem set we shall deal only with coherent input light, i.e. with light whose fluctuations are in the standard vacuum state.

(a) Show that the light beam's mean power — i.e. its power when one ignores the vacuum fluctuations — is

$$\bar{I} = \frac{\mathcal{A}}{8\pi} D^2 \; . \tag{3}$$

Here and throughout we use Gaussian units, not SI units.

(b) Assume that this mean power is large compared to the fluctuations. Then the fluctuations of the power $I(t)$ arise from a beating of the classical light amplitude D against the vacuum fluctuations $E_1^{\rm vac}$. Derive an equation for the spectral density of these power fluctuations $S_I(f)$ in terms of the mean power $\bar{I}$, the energy of a carrier photon $\hbar\omega_0$, and the spectral density of the dimensionless fluctuations $S_{a_1}(f)$.

(c) In his Lecture 20, Kip derived the spectral density of the power fluctuations $S_I(f)$ relying on two simple features of the light: that it consists of photons with individual energies $\hbar\omega_0$, and that these photons arrive randomly, in an uncorrelated way. These two properties, in fact, are the defining measurable features of light that is in a quantum mechanical coherent state. From Kip's result,

$$S_I(f) = 2\bar{I}\hbar\omega_0 \tag{4}$$

show that the dimensionless amplitude a_1 has unit spectral density. If the light had been excited in the $\sin[\omega(t - z/c)]$ quadrature, then this argument would have implied a unit spectral density for a_2. Therefore, for any light beam in a coherent state,

$$S_{a_1}(f) = S_{a_2}(f) = 1 \; . \tag{5}$$

(d) To make contact with the lecture of Buonanno and Chen, and with the reading for this week, Fourier analyze the dimensionless vacuum fluctuation operators:

$$a_j(t) = \int_{-\infty}^{\infty} a_j(\Omega) e^{-i\Omega t} \frac{d\Omega}{2\pi} \; . \tag{6}$$

[Note that $\Omega/2\pi = f$ is the side-band frequency, since in the electric field $a_j(t)$ is multiplied by the cosine or sine of $\omega_0 t$ in Eq. (1).] Since $a_j(t)$ is real, $a_j(-\Omega)$ is the complex conjugate (or, quantum mechanically, the Hermitian conjugate) of $a_j(+\Omega)$. This dictates the more conventional way of writing the Fourier transform (7):

$$a_j(t) = \int_0^\infty \left[a_j(\Omega)e^{-i\Omega t} + a_j^\dagger(\Omega)e^{+i\Omega t}\right] \ . \tag{7}$$

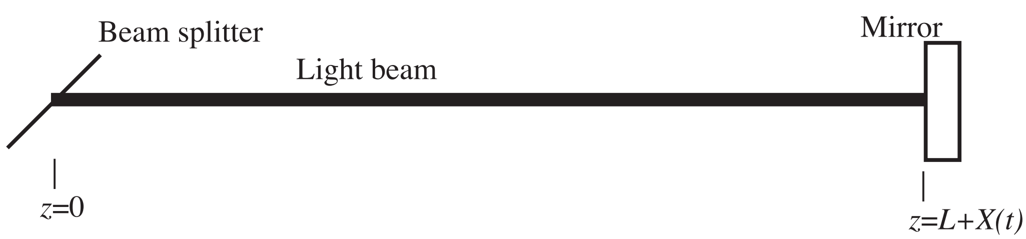
In the Buonanno-Chen reading for this week, the spectral density of $a_j(t)$ (or of any other quantity) is expressed in terms of its Fourier transform by the following relation:

$$\pi\delta(\Omega - \Omega')S_{a_j}(\Omega) = \langle\psi|a_j(\Omega)a_j^\dagger(\Omega')|\psi\rangle \ , \tag{8}$$

where $|\psi\rangle$ is the quantum state of the vacuum fluctuations. An equation analogous to this was encountered in your reading in Week 11(Eq. (5.46) of Blandford and Thorne, Ref. [6]), where it was a consequence of the Wiener-Khintchine theorem. From that Blandford-Thorne equation, derive Eq. (8). We shall not need Eq. (8) in this problem set.

2. **Quantum-optical noise in one arm of a simple GW interferometer**

Slides 7–12 of the Buonanno-Chen Lecture 24 sketch the derivation of the optical noises (shot noise and radiation pressure noise) in a conventional GW interferometer (e.g., LIGO-I) *without corner mirrors* (input mirrors). Here in this exercise, we shall explore some details of that derivation. To simplify the derivation, we focus on just one of the interferometer's arms, which has length L and has a perfectly reflecting mirror with mass m at its end as shown below:



(a) The input light is propagating in the $+z$ direction. At $z = 0$ (the beam splitter's location) it has the form explored in Exercise 1:

$$E^{\rm in}(t) = [D + E_1^{\rm vac}(t)]\cos\omega_0 t + E_2^{\rm vac}(t)\sin\omega_0 t \, , \tag{9}$$

where D is the (absolutely constant) carrier amplitude, and $E_1^{\rm vac}$ and $E_2^{\rm vac}$ are the vacuum fluctuations. Show that the output light that arrives back at $z = 0$ at time t has actually traveled a length $2L + 2X(t - L/c)$ inside the arm (where $X(t')$ is the gravity-wave-induced displacement of the mirror at time t'), and that therefore

$$\begin{aligned} E^{\rm out}(t) &= E^{\rm in}\left[t - 2L/c - 2X(t - L/c)\right] \\ &\approx [D + E_1^{\rm vac}(t - 2L/c)]\cos\omega_0 t \\ &\quad + \left[E_2^{\rm vac}(t - 2L/c) + \frac{2\omega_0 D}{c}X(t - L/c)\right]\sin\omega_0 t \ . \end{aligned} \tag{10}$$

(Here and throughout we use the fact that the mirror's speed $\dot{X}$ is small compared to the speed of light.) Estimate the fractional error made in the "≈"' of this equation. (It is an exceedingly small error). Notice that the GW signal (the X term) modulates the phase of the carrier. This phase is also modulated by the $E_2^{\rm vac}$ term, which thereby gives rise to the shot noise in the gravitational-wave signal. Using Eq. (10), explain why the amplitude of that shot noise in the GW signal will be inversely proportional to $\sqrt{\bar{I}_{\rm arm}}$, where $\bar{I}_{\rm arm}$ is the mean power of the light in the arm; i.e., the spectral density of the GW shot noise will be inversely proportional to $\bar{I}_{\rm arm}$.

(b) The light beam exerts a radiation-pressure force on the mirror. Explain why this radiation-pressure force has the magnitude $F = 2I_{\rm arm}/c$, where $I_{\rm arm}$ is the full light power (including shot-noise fluctuations) in the interferometer arm. As in Exercise 1, the mean light power is $\bar{I}_{\rm arm} = D^2\mathcal{A}/8\pi$ so the mean force is $\bar{F} = D^2\mathcal{A}/4\pi c$. This mean force is balanced by a steady counter force applied by the wires or fibers from which the mirror hangs, and it does not interest us. Rather, we are interested in the fluctuating part of this radiation-pressure force — the part produced by a beating of the vacuum fluctuations against the classical electric field. Show that this fluctuating force is given by

$$F_{\rm BA}(t) = \frac{D\mathcal{A}E_1^{\rm vac}(t-L/c)}{2\pi} \,. \tag{11}$$

The subscript "BA" arises from a valuable viewpoint on the measurement process: The light is being used to measure the mirror position, and in the act of making its measurement the light exerts the *back-action force* $F^{\rm BA}(t)$ on the measured object, the mirror. The mirror's position X evolves, in response to the combined influence of the gravitational waves and this back-action force, as follows:

$$X(t) = \frac{1}{2}Lh_{\rm GW}(t) + X_{\rm BA}(t) \qquad \ddot{X}_{\rm BA} = \frac{F_{\rm BA}}{m} \,. \tag{12}$$

Show that the mirror's "back-action noise" $X_{\rm BA}(t)$ is proportional to $\sqrt{\bar{I}_{\rm arm}}$.

(c) From Eqs. (10), (11) and (12), we can derive an *input-output relation* for the light in the arm. This is usually done in the frequency domain. Using the same decomposition as in Exercise 1, we write the input field (at the input point $z = 0$) in the following form:

$$E^{\rm in}(t) = [D + E_1(t-z/c)]\cos[\omega_0(t-z/c)] + E_2(t-z/c)\sin[\omega_0(t-z/c)] \,, \tag{13}$$

where

$$E_{1,2}(t) = \sqrt{\frac{4\pi\hbar\omega_0}{\mathcal{A}c}}\int_0^{+\infty}\frac{d\Omega}{2\pi}\left(a_{1,2}^{\rm arm}e^{-i\Omega t} + a_{1,2}^{\rm arm\,\dagger}e^{i\Omega t}\right) \;; \tag{14}$$

cf. Eqs. (1), (2), and (7). We write $E^{\rm out}(t)$ at $z = 0$ in this same form but with the arm's input dimensionless amplitudes $a_{1,2}^{\rm arm}$ replaced by output dimensionless amplitudes denoted $b_{1,2}^{\rm arm}$. Show that

$$\begin{aligned} b_1^{\rm arm} &= e^{2i\beta}a_1^{\rm arm} \\ b_2^{\rm arm} &= e^{2i\beta}\left(a_2^{\rm arm} - \mathcal{K}a_1^{\rm arm}\right) + \sqrt{2\mathcal{K}}\frac{h_{\rm GW}}{h_{\rm SQL}}e^{i\beta} \,, \end{aligned} \tag{15}$$

where

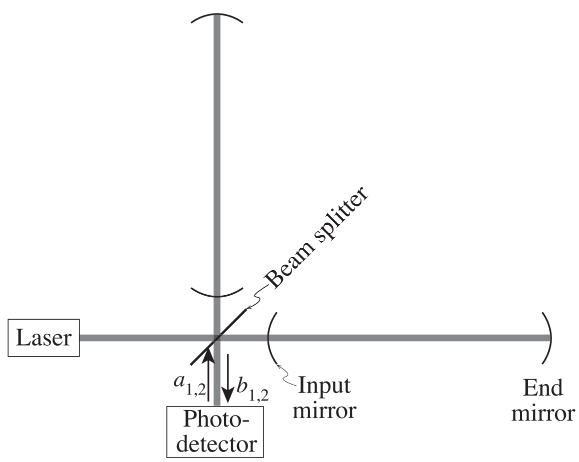
$$\beta = \frac{\Omega L}{c}\,, \quad \mathcal{K} = \frac{8\bar{I}_{\rm arm}\omega_0}{m\Omega^2 c^2}\,, \quad h_{\rm SQL} = \sqrt{\frac{8\hbar}{m\Omega^2 L^2}}\,. \tag{16}$$

(d) The amplitude of the *phase quadrature* $b_2^{\rm arm}$ is measured (e.g. by recombining the two arms' light in the beam splitter and performing photodetection in the manner of Exercise 5 of Week 11). Identify, in this measured quantity, the terms that represent the GW signal, the shot noise and the radiation pressure noise.

We could easily compute and study the spectral densities of the shot noise and radiation-pressure noise, but we shall not do so, since this interferometer is actually too idealized to be of practical interest. Instead, we shall compute those spectral densities for a conventional interferometer *with corner mirrors* such as LIGO-I (next exercise).

3. **Quantum optical noise in a conventional interferometer and the standard quantum limit (SQL)**

In a real, conventional interferometer such as LIGO-I, the arms are Fabry Perot cavities with corner ("input") mirrors, and the light from the two arms is recombined in the beamsplitter in the standard manner depicted in Slide 14 of the Buonanno-Chen lecture and in the following drawing:



(a) Such an interferometer is operated with mirror positions such that the laser light (almost) all returns toward the laser; almost none goes toward the photodetector (toward the "dark port"). Explain why this means that vacuum fluctuations, with side-band frequencies $f = \Omega/2\pi$ in the GW band behave as follows: Those that accompany the laser light into the interferometer from the bright port (almost) all return out the bright port, and those that enter from the dark port (almost) all return out the dark port. Since the GW signal is measured by the photodetector at the dark port, this means that the origin of the vacuum fluctuations that accompany the signal is the dark port. We denote those input vacuum fluctuations by $a_{1,2}$ and denote the output dimensionless electric field amplitudes at the dark port by $b_{1,2}$; see the figure above. These dimensionless amplitudes are related to the dark-port input and output electric fields by the analog of Eqs. (13) and (14).

(b) It turns out that the input-output relations for this conventional interferometer have the same form (15) as those for a single arm without a corner mirror, but with the quantities $\mathcal{K}$ and β modified due to the storage of the light in the arms' Fabry-Perot cavities:

$$b_1(\Omega) = a_1(\Omega)\, e^{2i\beta}, \qquad b_2(\Omega) = [a_2(\Omega) - \mathcal{K}\, a_1(\Omega)]\, e^{2i\beta} + \sqrt{2\mathcal{K}}\, \frac{h_{\rm GW}}{h_{\rm SQL}}\, e^{i\beta}\,, \quad (17)$$

with

$$\mathcal{K} = \frac{8 I_o \omega_o}{\Omega^2\,(\gamma^2+\Omega^2) m L^2}, \qquad h^2_{\rm SQL} = \frac{8\hbar}{m\Omega^2 L^2}, \qquad \gamma = \frac{Tc}{4L}, \qquad 2\beta = 2\arctan\frac{\Omega}{\gamma}, \quad (18)$$

where I_o is the laser power entering the beamsplitter, ω_o is the laser frequency, m is the mirror mass, L is the length of the arm cavities, T is the power transmissivity of the input mirrors and Ω is the sideband frequency. Explain in intuitive terms why the modifications of Eqs. (16) have the form (18); most especially, explain the origin of the factor $1/(\gamma^2+\Omega^2)$ in the expression for $\mathcal{K}$.

(c) In a conventional interferometer the quadrature phase $E_2^{\rm out}$ of the output electric field is measured (e.g. in the manner analyzed in Exercise 5 of Week 11). This is equivalent to measuring the dimensionless output field $b_2(t)$, which contains the GW signal $h_{\rm GW}(t)$. The Fourier transform of this output field is given by the input-output relation (17). Renormalize this measured field in such a way that the coefficient in front of $h_{\rm GW}(\Omega)$ is unity. Then the other terms represent the noise h_n that contaminates the measurement of $h_{\rm GW}$. Show that the Fourier transform of this GW noise is

$$h_n = \frac{h_{\rm SQL} e^{i\beta}}{\sqrt{2}} \left(\frac{-a_1}{\sqrt{\mathcal{K}}} + \sqrt{\mathcal{K}}\, a_2 \right)\,. \quad (19)$$

The input vacuum fluctuations $a_1(t)$ and $a_2(t)$ are in the standard quantum state, since nothing special is done in a conventional interferometer to put them into any other state. Therefore, they have unit spectral densities (Exercise 1). Show that the spectral density of the GW noise is given by

$$S_h(f) = \frac{h^2_{\rm SQL}}{2} \left(\frac{1}{\mathcal{K}} + \mathcal{K} \right)\,. \quad (20)$$

Which term in this spectral density is due to photon shot noise, and which term is due to radiation pressure noise?

(d) Plot this spectral density for various choices of the laser power I_o. On your plots indicate the shot noise and the radiation-pressure noise. Show that the sum of the two noises is always limited by the standard quantum limit (SQL)

$$S_h^{\rm SQL} = h^2_{\rm SQL} = \frac{8\hbar}{m\Omega^2 L^2}\,. \quad (21)$$

4. **Beating the SQL via the "variational output" technique**

Vyatchanin, Matsko and Zubova have invented a method to beat the SQL that entails nothing more than a change in the interferometer's readout, and Kimble et. al. [5] have devised a practical way of implementing their so-called "variational-output" interferometer.

(a) The Vyatchanin-Matsko-Zubova idea is to measure an appropriate mixture

$$b_\zeta = b_1 \sin\zeta + b_2 \cos\zeta \tag{22}$$

of the output electric field's amplitude and phase quadratures b_1 and b_2. Can you figure out a way to do this, when the chosen phase ζ is independent of sideband frequency? [Hint: try mixing the output light with another, reference light beam and then performing photodetection. How can this mixing be achieved? This technique is called "homodyne detection".]

(b) In a variational-output interferometer one performs this homodyne detection, but with a homodyne phase angle that depends on the (GW) sideband frequency, $\zeta = \zeta(\Omega)$. Kimble's key idea was that this can be done, in practice, by sending the output light through appropriate filters and then performing conventional homodyne detection (at fixed homodyne angle). Derive the Fourier transform of the GW noise $h_n(\Omega)$ [the analog of Eq. (19)] that plagues this measurement of b_ζ.

(c) Derive the spectral density of the GW noise $S_h(f)$ for this variational-output interferometer.

(d) Find the frequency-dependent form for ζ that minimizes $S_h(f)$, and compute the resulting minimized noise. Show that by this technique, at least in principle, the back-action noise can be removed completely from the GW signal. In practice, one's ability to do this is seriously limited by "optical losses" in the interferometer. This is because, in the measured quadrature b_ζ one is using correlations between the shot noise and the radiation-pressure noise to cancel out the radiation-pressure (back-action) noise; and optical losses (e.g. due to the scattering of light off mirrors, with vacuum leaking into the light beam to replace the scattered light) destroy those correlations. See Ref. [5].

(e) This is the first time we have met correlations in random processes. One can quantify the correlations using a quantity called the *cross spectral density*. Read about cross spectral densities in Section 5.3 of Blandford and Thorne, Ref. [6]. Then do the following: Identify in h_n [exercise 4b] the terms proportional to $\sqrt{I_o}$ as radiation-pressure noise and denote them by $[-4/(m\Omega^2)]\mathcal{F}$. [Here we are to think of $\mathcal{F}$ as the back-action force; $m/4$ is the reduced mass associated with the measured difference of the four mirror positions, and $m\Omega^2/4$ is the quantity that you multiply against the mirrors' position to get the force acting on the mirrors.] Identify the terms in h_n proportional to $1/\sqrt{I_o}$ as shot noise and denote them by $\mathcal{Z}$. Evaluate the spectral densities $S_\mathcal{F}$ and $S_\mathcal{Z}$ (denoted $S_{\mathcal{F}\mathcal{F}}$ and $S_{\mathcal{Z}\mathcal{Z}}$ by Buonanno and Chen), and the cross spectral densities $S_{\mathcal{Z}\mathcal{F}}$ and $S_{\mathcal{F}\mathcal{Z}}$. (In these evaluations, use the fact that for the standard vacuum state of the input fields $a_{1,2}$, there are no correlations between a_1 and a_2; i.e., their cross spectral densities vanish, $S_{a_1,a_2} = S_{a_2,a_1} = 0$.) Show that

$$S_\mathcal{Z}\, S_\mathcal{F} - S_{\mathcal{Z}\mathcal{F}}\, S_{\mathcal{F}\mathcal{Z}} = \hbar^2\,. \tag{23}$$

It turns out that the Heisenberg uncertainty principle, when applied to this measurement process, gaurantees that the quantity on the left can never be less than $\hbar^2$.

(f) Show that when $\zeta = 0$ (conventional interferometer) the correlation between shot noise and radiation-pressure noise is zero, i.e., $S_{\mathcal{Z}\mathcal{F}} = 0$. The resulting uncertainty relation $S_\mathcal{Z}\, S_\mathcal{F} \geq \hbar^2$ enforces the SQL.

5. **Optomechanical resonances in a signal recycled interferometer**

In this problem we shall derive the frequencies of the resonances in a signal recycled (SR) interferometer, in the limit of a perfectly reflecting signal-recycling mirror.

(a) Consider the propagation of a light beam over a distance l, from $z = 0$ to $z = l$. We know that

$$E(t, z = l) = E(t - l/c, z = 0)\,. \tag{24}$$

Decompose $E(t, z = l)$ and $E(t, z = 0)$ into quadrature fields and show that, the Fourier components of the quadrature fields, $a_{1,2}(\Omega)$ (at $z = 0$) and $b_{1,2}(\Omega)$ (at $z = l$), are related by the following rotation and phase shift

$$\begin{bmatrix} b_1(\Omega) \\ b_2(\Omega) \end{bmatrix} = e^{i\Phi} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} a_1(\Omega) \\ a_2(\Omega) \end{bmatrix} \tag{25}$$

where

$$\phi = \omega_0\, l/c\,, \quad \Phi = \Omega\, l/c\,. \tag{26}$$

(b) Consider the conventional interferometer in the figure above, and add a *perfectly reflecting* "signal recycling" mirror at the dark port. Assume that the distance from the dark port [the place where the input-output relations (17) are valid] to the mirror is l, with $\omega_0 l/c = 2N\pi + \phi$ and N an integer. The quantity ϕ is called the *detuning* phase. Assume also that l is so small that $\Omega l/c$ can be neglected for all frequencies Ω in the GW frequency band. Explain why the eigenfrequencies of the closed optical-mechanical system can be evaluated by imposing

$$\det\left[\mathbf{I} - \begin{pmatrix} \cos 2\phi & -\sin 2\phi \\ \sin 2\phi & \cos 2\phi \end{pmatrix} e^{2i\beta} \begin{pmatrix} 1 & 0 \\ -\mathcal{K} & 1 \end{pmatrix}\right] = 0\,, \tag{27}$$

where $\mathcal{K}$ and β are given by Eqs. (18). Show that Eq. (27) simplifies to the following eigenequation,

$$\cos(2\beta) - \cos(2\phi) - \frac{\mathcal{K}}{2}\sin(2\phi) = 0\,, \tag{28}$$

which can be further simplified (e.g. using Mathematica) to

$$\Omega^2(\Omega^2 - \gamma^2\tan^2\phi) + \frac{16\bar{I}_{\rm arm}\omega_0}{mL^2c}\gamma\tan\phi\,. \tag{29}$$

Here m is the mirror mass, c is the light speed, L is the arm-cavity length, and

$$\bar{I}_{\rm arm} = \frac{2}{T} I_0\,, \tag{30}$$

is the circulating power inside the arm cavities.

(c) Explain using Eq. (28) or Eq. (29) why, when the laser power is low, there is a pure optical resonance and a pure mechanical resonance. As I_0 gradually increases from 0, the two resonances become coupled and shifted from their original positions. In our simple example, the four roots of Eq. (29) can be derived analytically. Identify the regime where the system exhibits an optical spring behavior. In particular, what circulating power is required for this? In what region should the detuning-phase ϕ lie?