

**WEEK 13: Control Systems & Laser Locking;
Interferometer Simulation & Lock Acquisition; Seismic Isolation**

Lecture 23 by Eric Black [Control Systems & Laser Locking];

Lecture 24, Part 1 by Matt Evans [Interferometer Simulation & Lock Acquisition]

Lecture 24, Part 2 by Riccardo DeSalvo [Seismic Isolation]

Reading Related to These Lectures:

Items in bold are recommended; others are supplementary.

- 1. Laplace Transforms: Review Laplace transforms in whatever place you have studied them before, e.g. Mathews and Walker, Mathematical Methods of Physics.** [Comments: (i) Control theory is generally formulated in terms of Laplace transforms rather than Fourier transforms because Laplace transforms are more naturally suited to describing the transient response of a system to some input; the reason is that they entail only the behavior of the system between some initial time $t = 0$ and $t = \infty$, by contrast with Fourier transforms which involve the behavior over all time. In order to read control theory books and most any other engineering books about dynamics, it is necessary to understand Laplace transforms and their relation to Fourier transforms. (ii) Although theoretical physicists normally use the form $e^{-i\omega t}$ for the time dependence of a Fourier component of frequency ω , engineers and control theorists normally use $e^{+j\omega t}$ (where $i = j = \sqrt{-1}$).]
- 2. Control Systems: There are many books, and chapters in books, that treat control systems. The best one for you depends on your background, so instead of recommending specific ones, I shall give a few examples and leave it to you to choose which one is best for you.**
 - If you have no prior knowledge of control systems, I [Kip] suggest that you begin by reviewing the video of Eric Black's lecture (Lecture 23), and then work Exercises 1–4 below before reading any of the following references.
 - For a discussion of control systems in the context of interferometric gravitational-wave detectors such as LIGO, focusing especially on the control of the swinging of test masses in order to lock a Fabry Perot cavity, see Peter R. Saulson, *Fundamentals of Interferometric Gravitational Wave Detectors* (World Scientific, 1994), the last two pages of chapter 10, followed by chapter 11: “An overview of control theory”. [This material is on the course web site.] The discussion of Bode plots and the Nyquist criterion for stability in this reference is fairly minimal, so if you choose to read this, you may want also to read about those subjects in one of the following references.
 - A widely used text book on control theory, at the undergraduate level, is G.F. Franklin, J.D. Powell and A. Emami-Naeini, *Feedback Control of Dynamic Systems* (Prentice Hall, 2002). Bode Plots and the Nyquist criterion are discussed in the early parts of

Chapter 6. This book moves so slowly and covers things in so much detail, that it may be less appropriate for advanced physics students than more sophisticated texts that move more rapidly.

- d. An example of such a more sophisticated text is R.C. Dorf, *Modern Control Systems* (Addison-Wesley, various editions), and its sequel by R.C. Dorf and Richard H. Bishop. For Bode plots and the Nyquist criterion, see two chapters: “Frequency Response Methods” (chapter 7 in early editions; chapter 8 in later editions) and “Stability in the Frequency Domain” (chapter 8 in early editions; chapter 9 in later editions).
- e. A well written paper of historical interest is H.W. Bode, “Relations Between Attenuation and Phase in Feedback Amplifier Design,” *Bell Systems Technical Journal*, **19**, 421–454 (1940).

3. *Laser Locking (Pound-Drever-Hall Locking):*

- a. **E.D. Black**, “An introduction to Pound-Drever-Hall laser frequency stabilization,” *American Journal of Physics*, **69**, 79-87 (2001). Available on the course web site.
- b. R.W.P. Drever et. al., “Laser Phase Phase and Frequency Stabilization Using an Optical Resonator”, *Appl. Phys. B: Photophys. Laser Chem.*, **31**, 97-105 (1983). Available on the course web site.
- c. G.C. Bjorklund et al., “Frequency Modulation (FM) Spectroscopy: Theory of Line Shapes and Signal-to Noise Analysis,” *Appl Phys. B: Photophys. Laser Chem.* **32**, 145-152 (1983). Available on the course web site.

4. *Signal Extraction and Test-Mass Control in Interferometers.* Although these topics are not being covered in detail in any lectures, they are being referred to briefly by several lecturers and they are closely related to laser locking and to acquisition of lock. Here are some useful references:

- a. The following paper is a beautiful pedagogical treatment of signal extraction using RF modulation and demodulation; it also covers power recycling: **E.D. Black and R.N. Gutenkust**, “An Introduction to Signal Extraction in Interferometric Gravitational-Wave Detectors”, **working draft of a not-yet-submitted manuscript**. Available on the course web site.
- b. Here is a very brief paper that summarizes how signal extraction and test-mass control are achieved in LIGO: D. Sigg et. al., “Readout and Control of a Power-Recycled Interferometric Gravitational Wave Antenna,” LIGO document P-010038-00-D, available on the web at <http://www.ligo.caltech.edu/docs/P/P010038-00.pdf> .

5. *Interferometer modeling and acquisition of lock:*

- a. H. Yamamoto, B. Bhawal, M. Evans, E. Maros, M. Rakhmanov, and R.L. Savage, Jr., “End to End Simulation Program for Gravitational Wave Detectors”, LIGO Document number P000018-00-E, manuscript in preparation; available on the web at <http://www.ligo.caltech.edu/docs/P/P000018-00.pdf> .

- b. M. Evans et. al. “Lock Acquisition of a Gravitational Wave Interferometer”, *Optics Letters*, **27**, 598–600; available on the web at <http://www.ligo.caltech.edu/docs/P/P010015-04.pdf> .

This is a very brief summary of the lock acquisition system that Evans described in his lecture.

- c. M. Evans, *Lock Acquisition in Resonant Optical Interferometers*, unpublished PhD Thesis, Caltech (2001); available on the web at <http://www.ligo.caltech.edu/mevans/lockAcq/thesis.pdf> .

6. Seismic isolation:

- a. **Sections 8.1 – 8.7 of Peter R. Saulson, *Fundamentals of Interferometric Gravitational Wave Detectors* (World Scientific, 1994).** This is a very nice introduction to the subject.
- b. G. Losurdo et. al., “An Inverted Pendulum Preisolation Stage for the Virgo Suspension System,” *Review of Scientific Instruments*, **70**, 2507–2515 (1999).
- c. A. Bertolini, G. Cella, R. DeSalvo and V. Sanibale, “Seismic Noise Filtering, Vertical Resonance Frequency Reduction with Geometric Anti-springs: a Feasibility Study,” *Nuclear Instruments and Methods*, **A435**, 475–483 (1999).
- d. A. Bertolini et. al., “Recent Progress on the R&D Program for the Seismic Attenuation System (SAS) Proposed for the Advanced Gravitational Wave Detector, LIGO-II”, *Nuclear Instruments and Methods*, **A461**, 300–303 (2001).

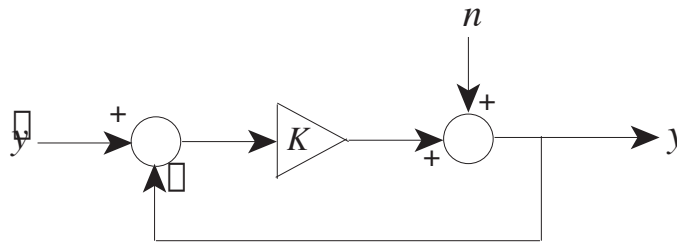
Assignment, to be turned in at beginning of class on Wednesday 24 April by students registered in the course:

- A. State what reading you have done, related to the course, during this past week.
- B. Work those exercises, from the list below, that are useful for you (i.e. that are at the appropriate level for you [neither much too hard nor too easy] and that have a ratio of grunge to learning that is reasonable.
- C. If A. and B. do not constitute enough to have taught you a reasonable amount about this week’s topic, then do one or more of the following:
- i. If you already know a lot about this week’s topic, just say so and stop.
 - ii. Invent your own exercises and work them.
 - iii. Carry out further reading and state what you have done.
 - iv. Seek private tutoring from a knowledgeable person about this week’s topic.
 - v. Pursue some other method of learning about this week’s topic, and state what you have done.

EXERCISES

A. Exercises that derive results quoted by Eric Black in his lecture on control systems, or illustrate principles in that lecture. The first three exercises were designed by him to mesh closely with his lecture; the fourth is an addendum by Kip.

1. Idealized Control System Subjected to Noise. Consider the control system depicted below; the input is \bar{y} , the output is y , and n is noise injected at the indicated point.



- a. In the absence of feedback, $y = K\bar{y} + n$. With feedback, show that the noise gets suppressed. Calculate the amount by which the noise is reduced. (Assume $K \gg 1$ and assume that the control system's responses are instantaneous, i.e. no time delays.)
- b. Now consider noise n injected before the amplifier K . Does this get suppressed by the feedback loop? Some components in a control loop must have intrinsically low noise, while others get their noise suppressed by feedback.

2. Connection between the Laplace transform and the physical behavior of an undriven system in the time domain. Denote by $L[f(t)] \equiv \int_0^\infty e^{-st} f(t) dt$ the Laplace transform of a function $f(t)$.

a. Show that

$$L[f'(t)] = sL[f(t)] - f(0) \tag{1}$$

and by extension that

$$L[f''(t)] = s^2L[f(t)] - sf(0) - f'(0) , \tag{2}$$

where the primes denote derivatives.

b. Consider a linear system governed by the differential equation

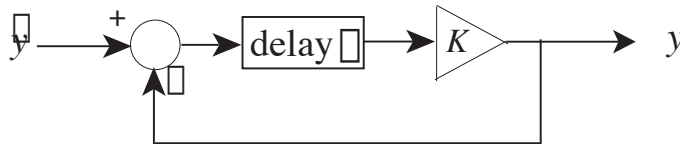
$$ay''(t) + by'(t) + cy(t) = x(t) . \tag{3}$$

Regard $x(t)$ as the input and $y(t)$ as the output, and let $H(s)$ be the transfer function for this linear system; i.e. $Y(s) = H(s)X(s)$, where $Y(s) = L[y(t)]$ and $X(s) = L[x(t)]$. Compute the transfer function. You can assume $y(0) = y'(0) = x(0) = 0$.

- c. What are the poles of this transfer function? These poles represent values of s at which the system can be excited without any input $x(t)$. Verify that this is the case by finding solutions with the form $y = Ae^{st}$ to Eq. (3) with $x = 0$. [Note: In the presence of input $x(t)$, one can compute $y(t)$ by inverting the Laplace transform. If you don't know how this is done, see a standard book on mathematical physics, e.g. Mathews and Walker, Ref. [1] above.

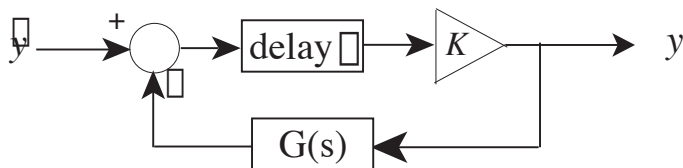
3. Instability of a Feedback Loop with Time Delay.

- a. Calculate the transfer function $D(s)$ corresponding to a time delay, $y(t) = x(t - \tau)$. [Hint: Evaluate $Y(s)$ in terms of $X(s)$ and take the ratio of the two expressions. You can assume $x(t) = 0$ for $t < 0$.]
- b. Consider the simple feedback loop with time delay that Black discussed in his lecture:



In reality, all amplifiers will display some time delay, so the time delay could be part of the amplifier. Evaluate the closed-loop transfer function $H(s) = Y(s)/\bar{Y}(s)$ for this feedback loop.

- c. This $H(s)$ has an infinite sequence of poles running along a line that is parallel to the imaginary s axis. Compute the values s_p of s at these poles (where p runs over $p = \pm 1, \pm 2, \pm 3, \dots$); and from them find the evolution $y(t) = \sum_p A_p e^{s_p t}$ in the absence of any input. Infer thereby that this feedback loop is stable for $K < 1$ and unstable for $K > 1$.
- d. Draw a Nyquist diagram for this feedback loop and use it to reproduce the result of part b: stability for $K < 1$ and instability for $K > 1$.
- e. Draw a Bode plot for this feedback loop and try to apply the Bode Gain-phase criterion for stability. This criterion fails. Explain why. What is it about real physical systems that typically prevents this kind of failure from happening?
4. **Stabilization of a Feedback Loop with Time Delay.** In order to make the above feedback loop force the output to follow the input, we must choose the amplification K large compared to unity, $K \gg 1$. However, for such K the loop will be unstable no matter how small the time delay τ . All physical systems will have some time delay, so such a feedback loop will necessarily be unstable. To stabilize it, we can modify the feedback signal by inserting a filter $G(s)$ into the loop:



Assume that the feedback loop is an electric circuit and $G(s)$ is a simple RC filter (a resistance R in the line and a capacitance C to ground). Then one can show using elementary circuit theory that $G(s) = 1/(1 + sT)$ where $T = RC$ is the filter's time constant. For concreteness set the gain of the amplifier to $K = 10$ and measure time in units of the time delay τ , so $\tau \equiv 1$.

- a. Draw a Nyquist diagram for various values of T (e.g., using Mathematica) and thereby deduce how large T must be (in units of τ) in order to stabilize the feedback loop. [Hint: It is only necessary to consider positive frequencies $\omega \geq 0$ in drawing the Nyquist diagram, since changing ω to $-\omega$ just reflects the Nyquist curve in the horizontal (real) axis.]
- b. Use a Bode plot to deduce how large T must be to stabilize the feedback loop.
- c. For what range of frequencies (in units of τ) does this feedback loop lock the output onto the input, to within, say ~ 30 per cent?

B. Exercises on Some Other Aspects of This Week's Material. Next week's exercises will also explore some of this week's material.

5. Modulated Light An electro-optic modulator (e.g. a PZT crystal) puts a radio-frequency modulation onto the laser light before it enters a gravitational-wave interferometer. The modulated light has phase $\omega_o t + \beta \sin \Omega t$, where $\omega_o \sim 10^{15} \text{s}^{-1}$ is the carrier frequency, t is time, β is the dimensionless amplitude of the modulation, and $\Omega \sim 10^8 \text{s}^{-1}$ is the modulation frequency. At fixed location the modulated light's electric field has the form $E = \Re[Ae^{i\omega_o t + i\beta \sin \Omega t}]$.

- a. From the "generating function" and associated series for Bessel functions, one can deduce that

$$e^{i\beta \sin \Omega t} = J_0(\beta) + \sum_{n=1}^{\infty} J_n(\beta) [e^{in\Omega t} + (-1)^n e^{-in\Omega t}] . \quad (4)$$

Use this to express the electric field E as a sum over contributions from the carrier and from sidebands at frequencies $\omega_o \pm \Omega, \omega_o \pm 2\Omega, \dots$.

- b. What fraction of the total power is in the carrier and what fraction in each side band, as a function of the modulation amplitude β ? Plot these fractions of power as a function of β for the carrier and the first few sidebands.

6. RF Readout for a Gravitational-Wave Interferometer

In Exercise 5 of Week 11 we studied a model of a GW interferometer in which the gravitational wave signal is extracted by beating its light (a ~ 100 Hz GW-induced side band of the carrier) against a bit of carrier light (frequency ω_o). This is a variant of what is called “homodyne detection” and is a possibility for LIGO-II. In LIGO-I and all other current interferometers the signal extraction is done differently: One beats the gravitational-wave signal against a radio-frequency sideband (frequency $\omega_o + \Omega t$ or $\omega_o - \Omega t$) and then demodulates to remove the radio frequency. This RF (radio-frequency) readout scheme is discussed in Ref. [4a] by Black and Gutenkunst. In this exercise we explore it a bit.

The RF modulation frequency Ω is chosen such that the sidebands $\omega_o \pm \Omega$ are well away from resonance for the interferometer’s arm cavities and so do not enter the cavities and do not pick up any gravitational-wave signal. The “Schnupp asymmetry” in the distances from the beam splitter to the two arm cavities is adjusted so that some desired portion $\epsilon^2 < 1$ of the RF sideband photons go out the interferometer’s “dark port”, toward the photodetector. The arm cavities and laser are adjusted via Pound-Drever-Hall locking so that the carrier frequency ω_o is precisely on resonance and enters the cavities and there acquires the gravitational-wave signal in the manner analyzed in Exercise 5 of Week 11.

- a. Explain in detail why the resulting light field going toward the photodetector has the following form:

$$E = \Re \left\{ \frac{1}{2} A J_0(\beta) e^{i\omega_o t} [e^{i\delta\phi} - e^{-i\delta\phi}] + \epsilon A J_1(\beta) e^{i\omega_o t} [e^{i\Omega t} - e^{-i\Omega t}] \right\}, \quad (5)$$

where A is the amplitude of the light impinging on the beam splitter, β is the amplitude of the phase modulation, and $\pm\delta\phi = \pm BkLh(t)$ is the phase shift put onto the light in the two arms [Eq. (4) of Week 11], with $k = \omega_o/c$ the wave number, L the interferometer arm length, $B = 4/(1 - R)$ the mean number of round trips that the light makes in an arm cavity, and R the reflectivity of the entrance mirror of the arm cavity. (As in Exercise 5 of Week 11, we assume for simplicity a gravitational-wave period long compared to the light storage time in the arm cavity.) We have ignored higher-order side bands, $n = 2, 3, 4, \dots$ since the demodulation process discards them and keeps only the first side bands $n = 1$.

- b. Show from Eq. (5) that the intensity of the light entering the photodetector is given by

$$I_{PD} = \bar{I}_{PD} + 2\sqrt{\bar{I}_{PD} I_o} J_0(\beta) BkLh(t) \sin \Omega t. \quad (6)$$

[There might be a factor ~ 2 error in this equation. Is there?] Here \bar{I}_{PD} is the power into the photodiode averaged over a time long compared to a carrier period but short compared to the RF period, and I_o is the laser power entering the interferometer at the beam splitter. Compare this with the corresponding result for homodyne readout, Eq. (7) of Week 11. Discuss the differences. Note that the second term, which carries the GW signal, has been generated by beating of the side-band electric field against the signal electric field, and note that this signal intensity is proportional to $J_0(\beta)$ but independent of $J_1(\beta)$. Why? Note further that the signal intensity (the second term) has the form of a slow (~ 100 Hz) amplitude modulation of a quantity that oscillates at a radio frequency ($\Omega/2\pi \sim 10$ MHz).

- b.** The light intensity I_{PD} is measured by the photodiode, producing a photocurrent that has a (nearly steady) component proportional to the first term of Eq. (6) and a slowly modulated radio-frequency component proportional to the second, signal term of Eq. (6). The gravitational-wave signal is read out by demodulating this photocurrent — i.e. by multiplying by $\sin \Omega t$ and averaging over times long compared to the RF period but short compared to GW periods. This demodulation leaves us with noise due to that portion of \bar{I}_{PD} that manages to get through the demodulation process (the shot-noise part of \bar{I}_{PD} that was in the immediate vicinity of the RF frequency $\Omega/2\pi$), plus the GW signal. Patterning your analysis after Exercise 5.b of Week 11, derive the spectral density of the GW shot noise $S_h(f)$ [analog of Eq. (9) of Week 11] for this RF readout scheme. Comment on the differences from the homodyne case.
- c.** One reason that this RF readout scheme is used in LIGO-I, instead of a homodyne readout scheme, is that the light intensity coming out of the LIGO-I lasers is much more noisy at frequencies ~ 100 Hz than simple shot-noise theory predicts (more noisy than the noise expected for a quantum mechanical coherent state of light), but at radio frequencies the laser noise is in accord with the shot-noise predictions for coherent light. Since the GW signal entering the photodetector is in sidebands of an RF-oscillating light intensity, it is the laser's shot noise at RF frequencies that contaminates the GW signal, not the shot noise at frequencies ~ 100 Hz.