Binary Black Holes: Numerical simulations and approximation methods

Lawrence E. Kidder
Cornell University

CaJAGWR Seminar
22 January 2008, Caltech

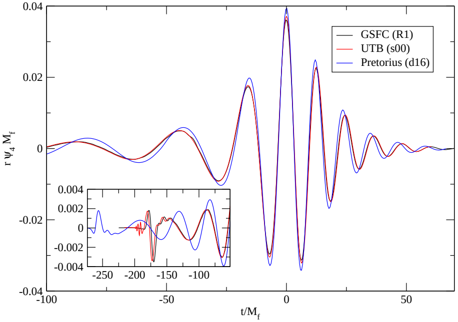
- Outline:
  - Simulations of binary black holes
  - Comparing NR vs PN waveforms
  - Final spin of the merged black hole
  - KITP workshop on NR and DA

BBH simulations: Methods

- Generalized Harmonic with Moving Excision [Pretorius 2005]
  - 10 second-order wave equations
  - constraint damping
  - Full adaptive mesh refinement
- Moving puncture [Campanelli et al 2006, Baker et al 2006]
  - BSSN formulation
  - Gauge driver conditions
  - Quickly adopted by multiple groups

NR BBH Results: Equal-mass, non-spinning

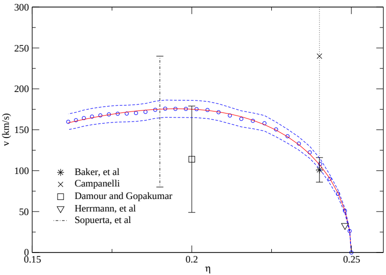
- Simple merger phase
  - Energy radiated $3.5 - 5\%$
  - Final spin $a/m = 0.69$
  - (2,2) mode dominates
- Progressed from
  1 orbit + merger + ringdown to
  16 orbits + merger + ringdown
  [Boyle et al 2007]
  [Caltech/Cornell 2008]
- Precision comparison with post-Newtonian
  (more details later in this talk)
- Initial comparison among codes
  [Baker, Campanelli, Pretorius, Zlochower 2007]



NR BBH Results: Unequal-mass, non-spinning

Maximum kick $175 \pm 11 km/s$
at mass-ratio 2.77

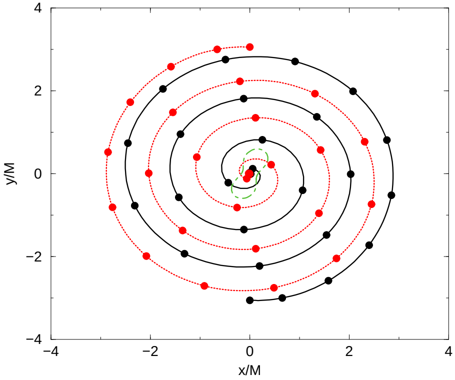
- Energy radiated decreases
- Final spin decreases
- Black hole kicks (random direction in orbital plane)
  - [Baker et al 2006]
  - [Herrmann et al 2007]
  - [González, et al 2007]
- Construction of templates [Ajith et al 2007]

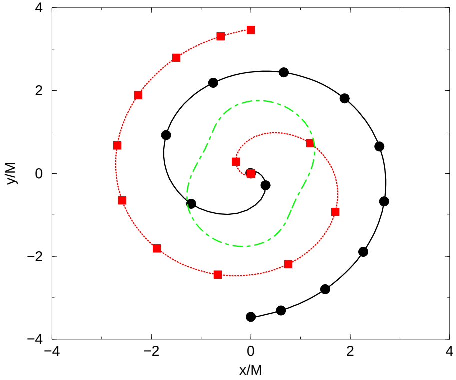


$v(km/s)$ vs
$\eta = (m_1 m_2)/(m_1 + m_2)^2$

NR BBH Results: Equal-mass, aligned spins

- Change in orbital dynamics (aligned vs anti-aligned spins) [Campanelli, Lousto, Zlochower 2006]
- Final spin aligned with orbital angular momentum (more later in talk)
- Kicks from unequal spins (up to $475 km/s$) [Herrmann et al 2007; Koppitz et al 2007]

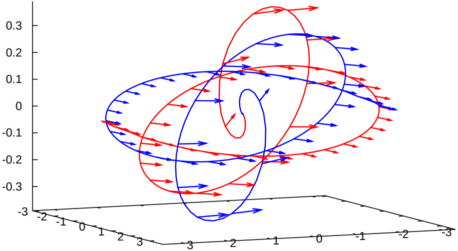




NR BBH Results: Equal-mass, precessing spins

- Precession of orbit and spins

[Campanelli, Lousto, Zlochower, Krishan, and Merritt, 2007]



- Super kicks! (up to $4000 km/s$) [Campanelli et al 2007; Gonzalez et al 2007]
- Kick can be perpendicular to orbital plane
- Fitting formula for kicks (e.g. [Herrmann et al 2007]) based on PN formula [Kidder 1995]

NR BH Results

- Generic binaries
  - Simulations get more expensive as mass ratio and spins increase
  - Expected to be big focus of 2008
- 3-body interactions [Campanelli, Lousto, Zlochower 2007]

Comparison between PN and NR

Collaborators at Cornell and Caltech:
Michael Boyle, Duncan A. Brown (LIGO), Gregory B. Cook (WFU), Lee Lindblom, Geoffrey Lovelace, Abdul H. Mroué, Robert Owen, Harald P. Pfeiffer, Oliver Rinne, Mark A. Scheel, Saul A. Teukolsky

- Compare Numerical (NR) and Post-Newtonian (PN) Waveforms
  - Test PN in late insprial
  - Test NR in regime where PN is valid
- Need long, accurate NR simulation

Spectral Einstein Code

[Kidder, Pfeiffer, Scheel]

- Generalized harmonic formulation of Einstein's equations
  - 10 coupled first-order wave equations (50 variables)
  - Constraint damping (Lindblom et al 2006)
- Dual frame method with dynamic tracking of the black holes
  - Time-dependent rotation and scaling (Scheel et al 2006)
  - Use control theory to adjust mapping to track holes
- Boundary conditions
  - Excision boundary is pure outflow (no BC needed)
  - Constraint preserving (Lindblom et al 2006)
  - No incoming physical radiation
  - Minimize reflections of gauge modes (Rinne et al 2007)
- Multidomain pseudospectral method
  - Exponential convergence for smooth solutions
  - Highly efficient for high accuracy

Spectral Einstein Code

[Kidder, Pfeiffer, Scheel]

- Generalized harmonic formulation of Einstein's equations
  - 10 coupled first-order wave equations (50 variables)
  - Constraint damping (Lindblom et al 2006)
- Dual frame method with dynamic tracking of the black holes
  - Time-dependent rotation and scaling (Scheel et al 2006)
  - Use control theory to adjust mapping to track holes
- Boundary conditions
  - Excision boundary is pure outflow (no BC needed)
  - Constraint preserving (Lindblom et al 2006)
  - No incoming physical radiation
  - Minimize reflections of gauge modes (Rinne et al 2007)
- Multidomain pseudospectral method
  - Exponential convergence for smooth solutions
  - Highly efficient for high accuracy

Spectral Einstein Code

[Kidder, Pfeiffer, Scheel]

- Generalized harmonic formulation of Einstein's equations
  - 10 coupled first-order wave equations (50 variables)
  - Constraint damping (Lindblom et al 2006)
- Dual frame method with dynamic tracking of the black holes
  - Time-dependent rotation and scaling (Scheel et al 2006)
  - Use control theory to adjust mapping to track holes
- Boundary conditions
  - Excision boundary is pure outflow (no BC needed)
  - Constraint preserving (Lindblom et al 2006)
  - No incoming physical radiation
  - Minimize reflections of gauge modes (Rinne et al 2007)
- Multidomain pseudospectral method
  - Exponential convergence for smooth solutions
  - Highly efficient for high accuracy

Spectral Einstein Code

[Kidder, Pfeiffer, Scheel]

- Generalized harmonic formulation of Einstein's equations
  - 10 coupled first-order wave equations (50 variables)
  - Constraint damping (Lindblom et al 2006)
- Dual frame method with dynamic tracking of the black holes
  - Time-dependent rotation and scaling (Scheel et al 2006)
  - Use control theory to adjust mapping to track holes
- Boundary conditions
  - Excision boundary is pure outflow (no BC needed)
  - Constraint preserving (Lindblom et al 2006)
  - No incoming physical radiation
  - Minimize reflections of gauge modes (Rinne et al 2007)
- Multidomain pseudospectral method
  - Exponential convergence for smooth solutions
  - Highly efficient for high accuracy

Extracting Gravitational Waves: Procedure

- Compute $\Psi_4 = -C_{\alpha\mu\beta\nu}\ell^\mu\ell^\nu\bar{m}^\alpha\bar{m}^\beta$ where $\ell^\mu \equiv \frac{1}{\sqrt{2}}(n^\mu - r^\mu)$, and m^μ is a complex null vector (satisfying $m^\mu\bar{m}_\mu = 1$)
  - Each extraction surface $\mathcal{E}$ is a coordinate sphere
  - n^μ is timelike unit normal to spatial hypersurface
  - r^μ is outward-pointing spatial unit normal to $\mathcal{E}$
  - $m^\mu = \frac{1}{\sqrt{2}r}\left(\frac{\partial}{\partial\theta} + i\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)^\mu$
  - Do not orthonormalize tetrad
- Concentrate on dominant $(2,2)$ mode

$$\Psi_4(t,r,\theta,\phi) = \sum_{lm} \Psi_4^{lm}(t,r)\,{}_{-2}Y_{lm}(\theta,\phi)$$

- Define gravitational wave amplitude A, phase ϕ and frequency ω

$$\Psi_4^{22}(r,t) = A(r,t)e^{-i\phi(r,t)}$$

$$\omega = \frac{d\phi}{dt}$$

Extrapolating Gravitational Waves: Procedure

- Extract at $r = 75m, 80m, 85m, \ldots, 235m, 240m$
- Shift each waveform by $r^* = r_{\rm areal} + 2M_{ADM} \log\left(\frac{r_{\rm areal}}{2M_{ADM}} - 1\right)$
- Fit phase and amplitude as polynomials:

$$\phi(t - r^*, r) = \phi^{(0)}(t - r^*) + \sum_{k=1}^{n} \frac{\phi^{(k)}(t - r^*)}{r^k}$$

$$rA(t - r^*, r) = A^{(0)}(t - r^*) + \sum_{k=1}^{n} \frac{A^{(k)}(t - r^*)}{r^k}$$

- Identify phase and amplitude at infinity with leading-order term:

$$\phi(t - r^*) = \phi^{(0)}(t - r^*)$$

and

$$rA(t - r^*) = A^{(0)}(t - r^*)$$

Post-Newtonian Approximation

(Blanchet, 2006, http://www.livingreviews.org/lrr-2006-4)

- Slow motion, weak field approximation $\epsilon \sim \frac{Gm}{rc^2} \sim (\frac{v}{c})^2$
- Equations of motion
  - Acceleration
  - Binding Energy E
  - Computed through $\epsilon^{7/2}$
  - Radiation reaction enters at $\epsilon^{5/2}$ and $\epsilon^{7/2}$
- Wave generation
  - Luminosity $\mathcal{L}$ computed through $\epsilon^{7/2}$
  - Polarization waveforms h_+ and $h_\times$ computed through $\epsilon^{5/2}$
- To produce accurate templates, assume:
  - Adiabatic inspiral of quasi-circular orbits
  - Energy balance

Post-Newtonian Approximation

(Blanchet, 2006, http://www.livingreviews.org/lrr-2006-4)

- Slow motion, weak field approximation $\epsilon \sim \frac{Gm}{rc^2} \sim (\frac{v}{c})^2$
- Equations of motion
  - Acceleration
  - Binding Energy E
  - Computed through $\epsilon^{7/2}$
  - Radiation reaction enters at $\epsilon^{5/2}$ and $\epsilon^{7/2}$
- Wave generation
  - Luminosity $\mathcal{L}$ computed through $\epsilon^{7/2}$
  - Polarization waveforms h_+ and $h_\times$ computed through $\epsilon^{5/2}$
- To produce accurate templates, assume:
  - Adiabatic inspiral of quasi-circular orbits
  - Energy balance

Post-Newtonian Approximation

(Blanchet, 2006, http://www.livingreviews.org/lrr-2006-4)

- Slow motion, weak field approximation $\epsilon \sim \frac{Gm}{rc^2} \sim (\frac{v}{c})^2$
- Equations of motion
  - Acceleration
  - Binding Energy E
  - Computed through $\epsilon^{7/2}$
  - Radiation reaction enters at $\epsilon^{5/2}$ and $\epsilon^{7/2}$
- Wave generation
  - Luminosity $\mathcal{L}$ computed through $\epsilon^{7/2}$
  - Polarization waveforms h_+ and $h_\times$ computed through $\epsilon^{5/2}$
- To produce accurate templates, assume:
  - Adiabatic inspiral of quasi-circular orbits
  - Energy balance

Post-Newtonian Approximation

(Blanchet, 2006, http://www.livingreviews.org/lrr-2006-4)

- Slow motion, weak field approximation $\epsilon \sim \frac{Gm}{rc^2} \sim (\frac{v}{c})^2$
- Equations of motion
  - Acceleration
  - Binding Energy E
  - Computed through $\epsilon^{7/2}$
  - Radiation reaction enters at $\epsilon^{5/2}$ and $\epsilon^{7/2}$
- Wave generation
  - Luminosity $\mathcal{L}$ computed through $\epsilon^{7/2}$
  - Polarization waveforms h_+ and $h_\times$ computed through $\epsilon^{5/2}$
- To produce accurate templates, assume:
  - Adiabatic inspiral of quasi-circular orbits
  - Energy balance

Adiabatic Inspiral of Quasicircular Orbits

- Emission of gravitational radiation circularizes the orbit
- Can express E, $\mathcal{L}$, h_+, and $h_\times$ in terms of an orbital frequency parameter $x \equiv (\frac{Gm\omega_{orb}}{c^3})^{2/3}$ and the orbital phase Φ
- For an equal mass, non-spinning binary

$$E = -\frac{mc^2}{8}x\left[1 - \frac{37}{48}x - \frac{1069}{384}x^2 + \left(\frac{1427365}{331776} - \frac{205}{384}\pi^2\right)x^3\right]$$

$$\begin{aligned}\mathcal{L} &= \frac{2c^5}{5G}x^5\left\{1 - \frac{373}{84}x + 4\pi x^{3/2} - \frac{59}{567}x^2 - \frac{767}{42}\pi x^{5/2}\right.\\ &\left.+\left[\frac{18608019757}{209563200} + \frac{355}{64}\pi^2 - \frac{1712}{105}\gamma - \frac{856}{105}\ln(16x)\right]x^3 + \frac{16655}{6048}\pi x^{7/2}\right\}\end{aligned}$$

Adiabatic Inspiral of Quasicircular Orbits

- Emission of gravitational radiation circularizes the orbit
- Can express E, $\mathcal{L}$, h_+, and $h_\times$ in terms of an orbital frequency parameter $x \equiv (\frac{Gm\omega_{orb}}{c^3})^{2/3}$ and the orbital phase Φ
- For an equal mass, non-spinning binary

$$E = -\frac{mc^2}{8}x\left[1 - \frac{37}{48}x - \frac{1069}{384}x^2 + \left(\frac{1427365}{331776} - \frac{205}{384}\pi^2\right)x^3\right]$$

$$\begin{aligned}\mathcal{L} &= \frac{2c^5}{5G}x^5\left\{1 - \frac{373}{84}x + 4\pi x^{3/2} - \frac{59}{567}x^2 - \frac{767}{42}\pi x^{5/2}\right.\\ &\left.+\left[\frac{18608019757}{209563200} + \frac{355}{64}\pi^2 - \frac{1712}{105}\gamma - \frac{856}{105}\ln(16x)\right]x^3 + \frac{16655}{6048}\pi x^{7/2}\right\}\end{aligned}$$

Adiabatic Inspiral of Quasicircular Orbits

- Emission of gravitational radiation circularizes the orbit
- Can express E, $\mathcal{L}$, h_+, and $h_\times$ in terms of an orbital frequency parameter $x \equiv (\frac{Gm\omega_{orb}}{c^3})^{2/3}$ and the orbital phase Φ
- For an equal mass, non-spinning binary

$$E = -\frac{mc^2}{8}x\left[1 - \frac{37}{48}x - \frac{1069}{384}x^2 + \left(\frac{1427365}{331776} - \frac{205}{384}\pi^2\right)x^3\right]$$

$$\begin{aligned}\mathcal{L} &= \frac{2c^5}{5G}x^5\left\{1 - \frac{373}{84}x + 4\pi x^{3/2} - \frac{59}{567}x^2 - \frac{767}{42}\pi x^{5/2}\right. \\ &\left. + \left[\frac{18608019757}{209563200} + \frac{355}{64}\pi^2 - \frac{1712}{105}\gamma - \frac{856}{105}\ln(16x)\right]x^3 + \frac{16655}{6048}\pi x^{7/2}\right\}\end{aligned}$$

PN waveforms: polarization waveforms and modes

Decompose polarization waveforms into spin-weighted spherical harmonics:

$$h_+ - ih_\times = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} h^{\ell m} {}_{-2}Y^{\ell m}(\Theta, \Phi)$$

Polarization waveforms known to 2.5 PN order [Arun et al 2004]

$$\begin{aligned} h_{ij}^{TT} &= \frac{4G}{c^2 R} \Pi_{ijmn} \sum_{\ell=2}^{\infty} \left\{ \frac{1}{c^\ell \ell!} \mathcal{U}_{mnL-2}(T_R) N_{L-2} \right. \\ &\left. + \frac{2\ell}{c^{\ell+1}(\ell+1)!} \epsilon_{pq(m} \mathcal{V}_{n)pL-2}(T_R) N_{qL-2} \right\}. \end{aligned}$$

Can compute modes directly from multipole moments! [Kidder 2008]

$$h^{\ell m} = \frac{G}{\sqrt{2} R c^{\ell+2}} \left(U^{\ell m}(T_R) - \frac{i}{c} V^{\ell m}(T_R) \right)$$

PN waveforms: modes to full known order

Radiative Multipole	PN order for $h_{+,\times}$	Known PN order
$\mathcal{I}_{ij}$	2.5	3
$\mathcal{J}_{ij}$	2	2
$\mathcal{I}_{ijk}$	2	2
$\mathcal{J}_{ijk}$	1.5	1.5
$\mathcal{I}_{ijkl}$	1.5	2
$\mathcal{J}_{ijkl}$	1	1
$\mathcal{I}_{ijklm}$	1	1
$\mathcal{J}_{ijklm}$	0.5	1
$\mathcal{I}_{ijklmn}$	0.5	1
$\mathcal{J}_{ijklmn}$	0	1
$\mathcal{I}_{ijklmno}$	0	1
$\mathcal{I}_L(\ell > 7)$	-	1
$\mathcal{J}_L(\ell > 6)$	-	1

PN waveforms: modes to full known order

- For an equal mass, non-spinning binary

$$
\begin{aligned}
(h_+ - ih_\times)^{(2,2)} \quad = \quad & -2\sqrt{\frac{\pi}{5}}\frac{Gm}{c^2 R}e^{-2i\Phi}x\left\{1 - \frac{373}{168}x + 2\pi x^{3/2} - \frac{62653}{24192}x^2\right. \\
& - \left[\frac{197}{42}\pi + 6i\right]x^{5/2} + \left[\frac{43876092677}{1117670400} + \frac{99}{128}\pi^2\right. \\
& \left.\left. - \frac{428}{105}\ln x - \frac{856}{105}\gamma - \frac{1712}{105} + \frac{428}{105}i\pi\right]x^3\right\}
\end{aligned}
$$

[Kidder, 2008]8

Energy Balance

$$\frac{dE}{dt} = -\mathcal{L}$$

- Given $E(x)$, $\mathcal{L}(x)$ and $h(x)$ need to choose:
  - Order k_Φ through which terms in E and $\mathcal{L}$ are kept (up to 3.5)
  - Order k_A through which terms in h are kept (up to 2.5)
  - How to use energy balance to obtain $x(t)$ and $\Phi(t)$

$$\frac{dx}{dt} = -\frac{\mathcal{L}}{(dE/dx)}$$

Energy Balance

$$\frac{dE}{dt} = -\mathcal{L}$$

- Given $E(x)$, $\mathcal{L}(x)$ and $h(x)$ need to choose:
  - Order k_Φ through which terms in E and $\mathcal{L}$ are kept (up to 3.5)
  - Order k_A through which terms in h are kept (up to 2.5)
  - How to use energy balance to obtain $x(t)$ and $\Phi(t)$

$$\frac{dx}{dt} = -\frac{\mathcal{L}}{(dE/dx)}$$

Energy Balance

$$\frac{dE}{dt} = -\mathcal{L}$$

- Given $E(x)$, $\mathcal{L}(x)$ and $h(x)$ need to choose:
  - Order k_Φ through which terms in E and $\mathcal{L}$ are kept (up to 3.5)
  - Order k_A through which terms in h are kept (up to 2.5)
  - How to use energy balance to obtain $x(t)$ and $\Phi(t)$

$$\frac{dx}{dt} = -\frac{\mathcal{L}}{(dE/dx)}$$

Energy Balance

$$\frac{dE}{dt} = -\mathcal{L}$$

- Given $E(x)$, $\mathcal{L}(x)$ and $h(x)$ need to choose:
  - Order k_Φ through which terms in E and $\mathcal{L}$ are kept (up to 3.5)
  - Order k_A through which terms in h are kept (up to 2.5)
  - How to use energy balance to obtain $x(t)$ and $\Phi(t)$

$$\frac{dx}{dt} = -\frac{\mathcal{L}}{(dE/dx)}$$

PN Taylor Approximants

(Damour, Iyer, Sathyaprakash, 2001)

- TaylorT1: Numerically integrate $\frac{dx}{dt} = -\frac{\mathcal{L}}{(dE/dx)}$ and $\frac{d\Phi}{dt} = \frac{x^{3/2}c^3}{Gm}$
- TaylorT2: Analytically integrate $t(x) = t_0 + \int_x^{x_0} dx \frac{(dE/dx)}{\mathcal{L}}$ and $\Phi(x) = \Phi_0 + \int_x^{x_0} dx \frac{x^{3/2}c^3}{Gm}\frac{(dE/dx)}{\mathcal{L}}$
- TaylorT3: Introduce $\tau \equiv \frac{\nu c^3}{5Gm}(t_0 - t)$ where $\tau^{-1/4} = O(c^{-2})$ and invert the TaylorT2 expression for $t(x)$ to obtain $x(t)$ and then $\Phi(t)$

(Buonanno, Cook, Pretorius, 2007)

- TaylorT4: Expand $\mathcal{F}(x) = -\frac{\mathcal{L}}{(dE/dx)}$ as a single Taylor series and numerically integrate $\frac{dx}{dt} = \mathcal{F}$ and $\frac{d\Phi}{dt} = \frac{x^{3/2}c^3}{Gm}$

PN Taylor Approximants

(Damour, Iyer, Sathyaprakash, 2001)

- TaylorT1: Numerically integrate $\frac{dx}{dt} = -\frac{\mathcal{L}}{(dE/dx)}$ and $\frac{d\Phi}{dt} = \frac{x^{3/2}c^3}{Gm}$
- TaylorT2: Analytically integrate $t(x) = t_0 + \int_x^{x_0} dx \frac{(dE/dx)}{\mathcal{L}}$ and $\Phi(x) = \Phi_0 + \int_x^{x_0} dx \frac{x^{3/2}c^3}{Gm}\frac{(dE/dx)}{\mathcal{L}}$
- TaylorT3: Introduce $\tau \equiv \frac{\nu c^3}{5Gm}(t_0 - t)$ where $\tau^{-1/4} = O(c^{-2})$ and invert the TaylorT2 expression for $t(x)$ to obtain $x(t)$ and then $\Phi(t)$

(Buonanno, Cook, Pretorius, 2007)

- TaylorT4: Expand $\mathcal{F}(x) = -\frac{\mathcal{L}}{(dE/dx)}$ as a single Taylor series and numerically integrate $\frac{dx}{dt} = \mathcal{F}$ and $\frac{d\Phi}{dt} = \frac{x^{3/2}c^3}{Gm}$

PN Taylor Approximants

(Damour, Iyer, Sathyaprakash, 2001)

- TaylorT1: Numerically integrate $\frac{dx}{dt} = -\frac{\mathcal{L}}{(dE/dx)}$ and $\frac{d\Phi}{dt} = \frac{x^{3/2}c^3}{Gm}$
- TaylorT2: Analytically integrate $t(x) = t_0 + \int_x^{x_0} dx \frac{(dE/dx)}{\mathcal{L}}$ and $\Phi(x) = \Phi_0 + \int_x^{x_0} dx \frac{x^{3/2}c^3}{Gm}\frac{(dE/dx)}{\mathcal{L}}$
- TaylorT3: Introduce $\tau \equiv \frac{\nu c^3}{5Gm}(t_0 - t)$ where $\tau^{-1/4} = O(c^{-2})$ and invert the TaylorT2 expression for $t(x)$ to obtain $x(t)$ and then $\Phi(t)$

(Buonanno, Cook, Pretorius, 2007)

- TaylorT4: Expand $\mathcal{F}(x) = -\frac{\mathcal{L}}{(dE/dx)}$ as a single Taylor series and numerically integrate $\frac{dx}{dt} = \mathcal{F}$ and $\frac{d\Phi}{dt} = \frac{x^{3/2}c^3}{Gm}$

PN Taylor Approximants

(Damour, Iyer, Sathyaprakash, 2001)

- TaylorT1: Numerically integrate $\frac{dx}{dt} = -\frac{\mathcal{L}}{(dE/dx)}$ and $\frac{d\Phi}{dt} = \frac{x^{3/2}c^3}{Gm}$
- TaylorT2: Analytically integrate $t(x) = t_0 + \int_x^{x_0} dx \frac{(dE/dx)}{\mathcal{L}}$ and $\Phi(x) = \Phi_0 + \int_x^{x_0} dx \frac{x^{3/2}c^3}{Gm} \frac{(dE/dx)}{\mathcal{L}}$
- TaylorT3: Introduce $\tau \equiv \frac{\nu c^3}{5Gm}(t_0 - t)$ where $\tau^{-1/4} = O(c^{-2})$ and invert the TaylorT2 expression for $t(x)$ to obtain $x(t)$ and then $\Phi(t)$

(Buonanno, Cook, Pretorius, 2007)

- TaylorT4: Expand $\mathcal{F}(x) = -\frac{\mathcal{L}}{(dE/dx)}$ as a single Taylor series and numerically integrate $\frac{dx}{dt} = \mathcal{F}$ and $\frac{d\Phi}{dt} = \frac{x^{3/2}c^3}{Gm}$

Comparing Waveforms: Procedure

- Equate PN total mass to twice the irreducible mass of each BH
- $\Psi_4^{PN} = \frac{\partial^2}{\partial t^2}(h_+ - ih_\times)$
- Compare the (2,2) components of Ψ_4^{NR} and Ψ_4^{PN}
- At a given matching frequency ω_m
  - Perform a time shift so $t_{NR}(\omega_m) = t_{PN}(\omega_m)$
  - Perform a phase shift so $\phi_{NR}(\omega_m) = \phi_{PN}(\omega_m)$

Comparing Waveforms: Procedure

- Equate PN total mass to twice the irreducible mass of each BH
- $\Psi_4^{PN} = \frac{\partial^2}{\partial t^2}(h_+ - ih_\times)$
- Compare the (2,2) components of Ψ_4^{NR} and Ψ_4^{PN}
- At a given matching frequency ω_m
  - Perform a time shift so $t_{NR}(\omega_m) = t_{PN}(\omega_m)$
  - Perform a phase shift so $\phi_{NR}(\omega_m) = \phi_{PN}(\omega_m)$

Comparing Waveforms: Procedure

- Equate PN total mass to twice the irreducible mass of each BH
- $\Psi_4^{PN} = \frac{\partial^2}{\partial t^2}(h_+ - ih_\times)$
- Compare the (2,2) components of Ψ_4^{NR} and Ψ_4^{PN}
- At a given matching frequency ω_m
  - Perform a time shift so $t_{NR}(\omega_m) = t_{PN}(\omega_m)$
  - Perform a phase shift so $\phi_{NR}(\omega_m) = \phi_{PN}(\omega_m)$

Comparing Waveforms: Procedure

- Equate PN total mass to twice the irreducible mass of each BH
- $\Psi_4^{PN} = \frac{\partial^2}{\partial t^2}(h_+ - ih_\times)$
- Compare the (2,2) components of Ψ_4^{NR} and Ψ_4^{PN}
- At a given matching frequency ω_m
  - Perform a time shift so $t_{NR}(\omega_m) = t_{PN}(\omega_m)$
  - Perform a phase shift so $\phi_{NR}(\omega_m) = \phi_{PN}(\omega_m)$

Comparing Waveforms: Procedure

- Equate PN total mass to twice the irreducible mass of each BH
- $\Psi_4^{PN} = \frac{\partial^2}{\partial t^2}(h_+ - ih_\times)$
- Compare the (2,2) components of Ψ_4^{NR} and Ψ_4^{PN}
- At a given matching frequency ω_m
  - Perform a time shift so $t_{NR}(\omega_m) = t_{PN}(\omega_m)$
  - Perform a phase shift so $\phi_{NR}(\omega_m) = \phi_{PN}(\omega_m)$

Comparing Waveforms: Procedure

- Equate PN total mass to twice the irreducible mass of each BH
- $\Psi_4^{PN} = \frac{\partial^2}{\partial t^2}(h_+ - ih_\times)$
- Compare the (2,2) components of Ψ_4^{NR} and Ψ_4^{PN}
- At a given matching frequency ω_m
    - Perform a time shift so $t_{NR}(\omega_m) = t_{PN}(\omega_m)$
    - Perform a phase shift so $\phi_{NR}(\omega_m) = \phi_{PN}(\omega_m)$

Comparing Waveforms: Estimated Errors

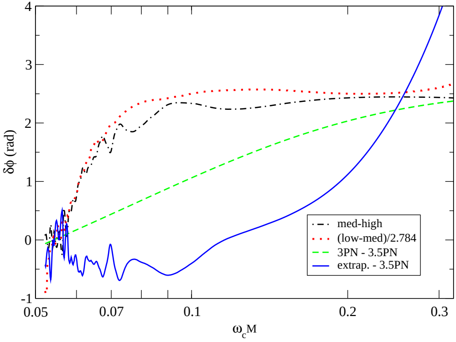
Effect	$\delta\phi$ (radians)	$\delta A/A$
Numerical truncation error	0.003	0.001
Finite outer boundary	0.005	0.002
Extrapolation $r \to \infty$	0.005	0.002
GW extraction at $r_{\rm areal}$=const?	0.002	10^{-4}
Drift of mass m	0.002	10^{-4}
Coordinate time = proper time?	0.002	10^{-4}
Lapse spherically symmetric?	0.01	$4 \cdot 10^{-4}$
residual eccentricity	0.02[1]	0.004
residual spins	0.03	0.001
root-mean-square sum	**0.04**[1]	**0.005**

[1] For matching at $m\omega_m = 0.04$, the phase error due to residual eccentricity increases to 0.05 radians, thus increasing the root-mean-square sum to 0.06 radians.

Comparing Waveforms: Goddard (gr-qc/0612024)

Consistency of post-Newtonian waveforms with numerical relativity
(Baker, van Meter, McWilliams, Centrella, Kelly)

- 7 orbits ($e < 0.01$)
- 4th-order AMR, puncture
- extract at $60M$
- TaylorT4
- Match at $m\omega = 0.054$
- $m\omega = 0.15$ is about 1.5 orbits to merger

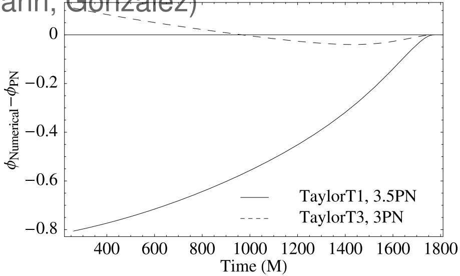


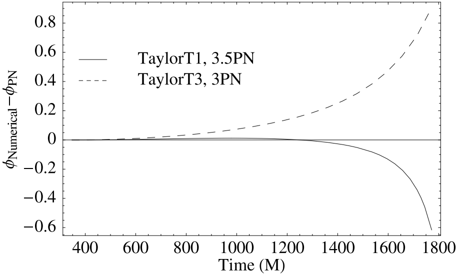
Comparing Waveforms: Jena (gr-qc/0706.1305v2)

Where post-Newtonian and numerical-relativity waveforms meet

(Hannam, Husa, Sperhake, Bruegmann, Gonzalez)

- 9 orbits ($e < 0.0016$)
- 6th-order AMR, puncture
- Extract at $(50M, 60M, 70M, 80M, 90M)$
- Extrapolate $A(\phi)$ to ∞
- phase accuracy $0.25 radians$
- amplitude accuracy 2%
- match at $m\omega = 0.1, 0.0455$

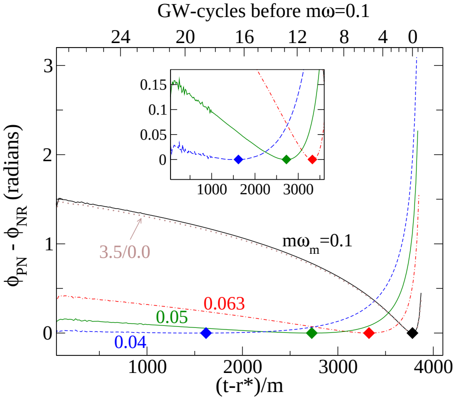


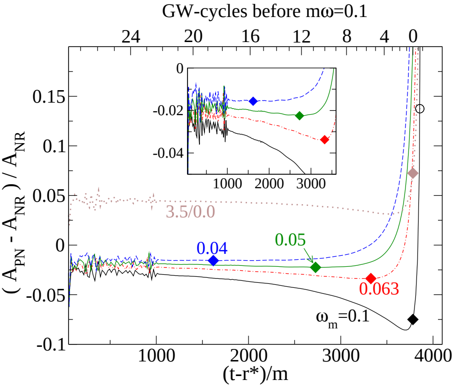


Comparing Waveforms: NR and PN Taylor T1

- Numerically integrate

$$\frac{dx}{dt} = -\frac{\mathcal{L}}{(dE/dx)} = \frac{16c^3}{5Gm}x^5 \frac{1 - \frac{373}{84}x + \cdots}{1 - \frac{37}{24}x + \cdots}$$

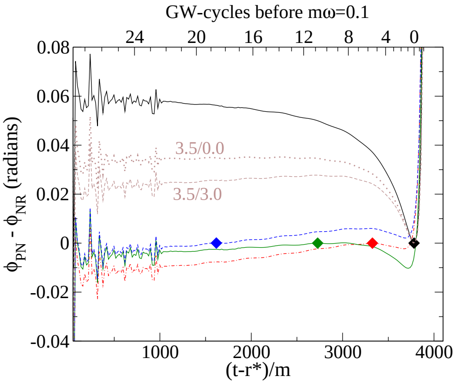


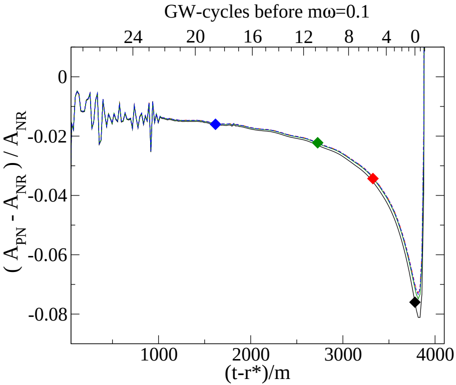


Comparing Waveforms: NR and PN Taylor T4

- Numerically integrate

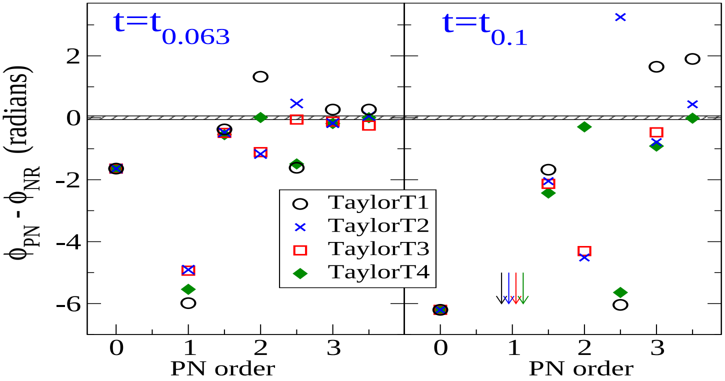
$$\frac{dx}{dt} = -\frac{\mathcal{L}}{(dE/dx)} \sim \frac{16c^3}{5Gm}x^5\left(1 - \frac{487}{168}x + \cdots\right)$$





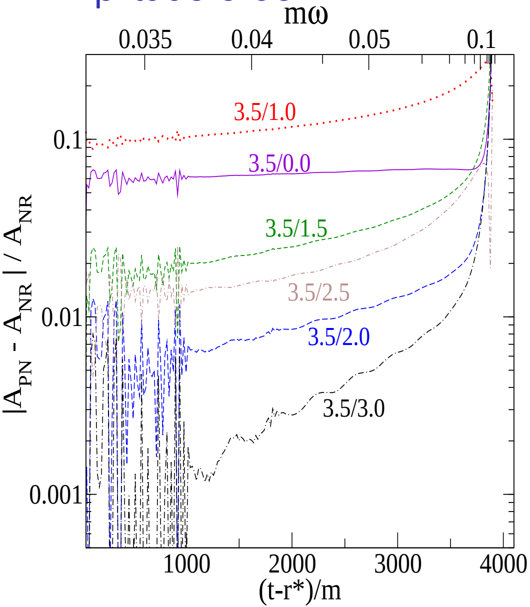
NR vs PN: Varying PN order

- matched at $m\omega = 0.04$



NR vs PN: Varying PN Amplitude order

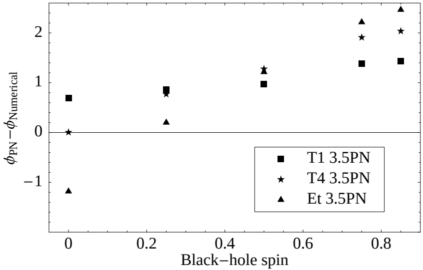
- Taylor T4 matched at $m\omega = 0.04$
- 3.5 PN in phase
- vary PN in amplitude
- (2,2) mode
- Suggests computing full 3PN amplitude



NR vs PN: Effects of spin

[Hannam, Husa, Bruegmann, Gopakumar, 2007]

- Spins aligned with orbital angular momentum.
- Spin-orbit effects through 2.5PN
- Spin-spin throught 2PN
- phase error in 10 cycles prior to $m\omega = 0.1$



Astrophysical parameters of the final black hole

- Given initial masses and spins predict:
  - Final mass
  - Final spin
  - Recoil velocity
- Develop formulae from:
  - Test-particle calculations
  - post-Newtonian
  - symmetry arguments
  - numerical simulations

Final Spin: Approach

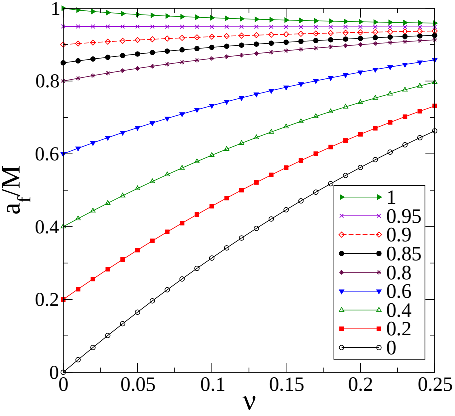
[Buonanno, Kidder, Lehner 2008]

- Assume:
  - mass of system is conserved
  - magnitude of individual spins remain constant
  - angular momentum radiated away from ISCO to merger is negligible
  - Choose ISCO of Kerr black hole with spin parameter of the *final* black hole

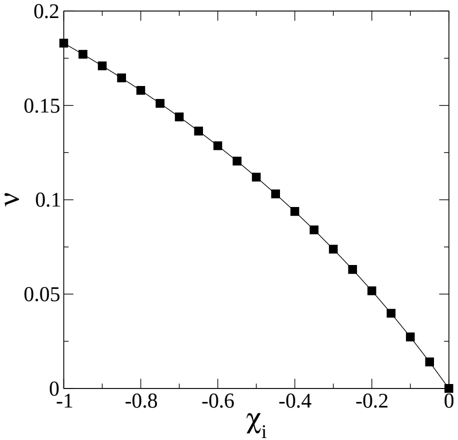
$$\frac{a_f}{M} = \frac{L_{\rm orb}}{M^2}(\mu, r_{\rm ISCO}, a_f) + \frac{m_1 a_1}{M^2} + \frac{m_2 a_2}{M^2}$$

Final Spin: Predictions

- Final spin for aligned equal spins as function of symmetric mass ratio
  $\nu = m_1 m_2/(m_1 + m_2)^2$

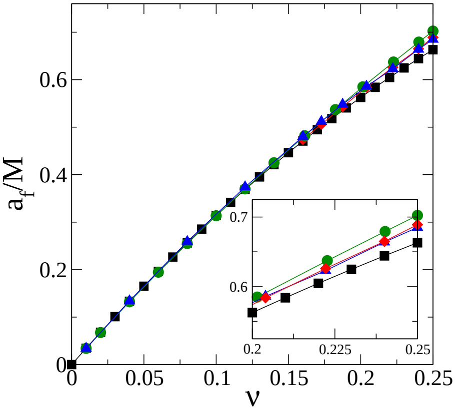


- Final non-spinning hole for equal spin parameters
  $\chi_i = a_i/m_i$
  (anti-aligned with $\vec{L}$)

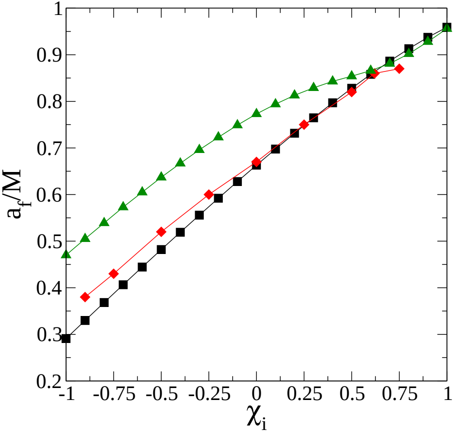


Final Spin: Comparison with numerical results

- Non-spinning binary
  Our prediction [BKL]
  EOB: [Damour Nagar 2007]
  NR: [Berti et al 2007]
  NR+TP: [Buonanno et al 2007]



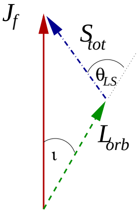
- Equal-mass, equal aligned spins
  Our prediction [BKL]
  EOB: [Buonanno et al 2006]
  NR: [Marronetti et al 2007]



Final Spin: Generic spins

- Assume magnitude of total spin and angle between total spin and direction of angular orbital momentum remain constant
- Solve system of equations

FAU : [Tichy and Marronetti 2007]
RIT : [Campanelli et al 2007]
PSU : [Herrmann et al 2007]

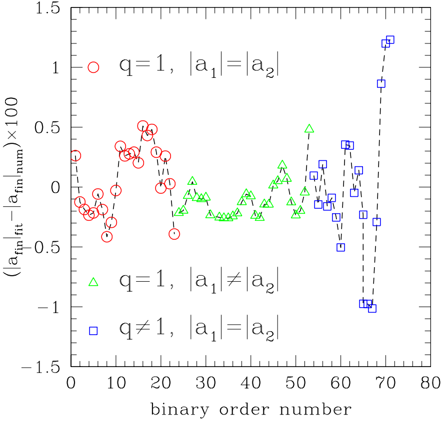


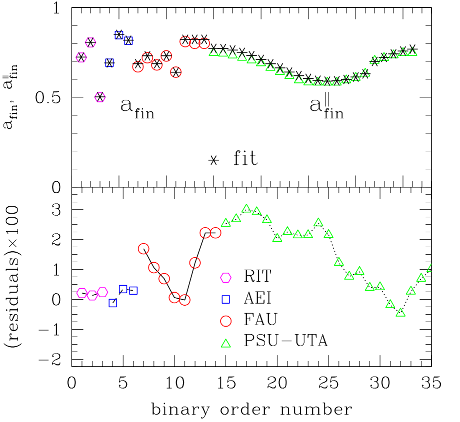
Case	a_f/M	a_f/M
C_1(FAU)	0.67	0.66
C_2(FAU)	0.72	0.71
C_3(FAU)	0.68	0.66
C_4(FAU)	0.73	0.71
C_5(FAU)	0.64	0.61
C_6(FAU)	0.81	0.82
C_7(FAU)	0.80	0.82
C_8(FAU)	0.80	0.82
SP_3(RIT)	0.72	0.70
SP_4(RIT)	0.81	0.80
SP_6(RIT)	0.50	0.48
$S-0$(PSU)	0.75	0.78
$S-75$(PSU)	0.69	0.69
$S-150$(PSU)	0.58	0.57
$S-225$(PSU)	0.61	0.64
$S-300$(PSU)	0.72	0.76

Final Spin: AEI simulations

[Rezzolla et al 2007]

- Developed formula for final spin
- Took ideas from [Bounanno, Kidder, Lehner, 2008]
- Fit coefficients from known simulations
- Needs testing in generic cases





KITP Miniprogram: Interplay between Numerical Relativity and Data Analysis

- Use match to assess waveforms for detection purposes
- For parameter estimation want systematic errors to be less than statistical errors. Match is not adequate for this purpose. Lot of work to do!
- Calibration errors (goal: few degrees in phase, 10% in amplitude)
- Need large number of short waveforms covering parameter space to make sure there are no holes in search.
- Need longer accurate waveforms to construct/calibrate/test waveform templates

Numerical INJection Analysis (NINJA) project

http://www.gravity.phy.syr.edu/dokuwiki/doku.php?id=ninja:home

- Goal is to test efficiency of various data analysis pipelines on numerical relativity wavefroms buried in Gaussian noise.
- All numerical relativists and data analysts are invited to participate

Summary

- Numerical relativity key to understanding BBH
- NR can be used to develop templates and fitting formulae
- NR needed to explore parameters and test analytic models